TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 349, Number 5, May 1997, Pages 1821–1850
S 0002-9947(97)01710-8

# THE HOMOTOPY GROUPS OF THE $L_2$-LOCALIZED TODA-SMITH SPECTRUM $V(1)$ AT THE PRIME 3

KATSUMI SHIMOMURA

Abstract. In this paper, we try to compute the homotopy groups of the $L_2$-localized Toda-Smith spectrum $V(1)$ at the prime 3 by using the Adams-Novikov spectral sequence, and have almost done so. This computation involves non-trivial differentials $d_5$ and $d_9$ of the Adams-Novikov spectral sequence, different from the case $p > 3$. We also determine the homotopy groups of some $L_2$-localized finite spectra relating to $V(1)$. We further show some of the non-trivial differentials on elements relating so-called $\beta$-elements in the Adams-Novikov spectral sequence for $\pi_*(S^0)$.

## Introduction

Let $L_n$ be the Bousfield localization functor from the category of spectra to itself with respect to $E(n)$ [19], and $V(k)$ denote the Toda-Smith spectrum [26] with $BP_*$-homology $BP_*/(p, v_1, \cdots, v_n)$, at each prime number $p$. Here $BP$ and $E(n)$ denote the Brown-Peterson and the Johnson-Wilson spectra with coefficient rings $BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \cdots]$ and $E(n)_* = \mathbf{Z}_{(p)}[v_1, \cdots, v_n, v_n^{-1}]$ with $|v_n| = 2(p^n - 1)$. The Toda-Smith spectrum $V(k)$ is known to exist for $k < 4$ if and only if $p > 2k$ [26], [20], and note that $V(-1) = S^0$ and $V(0)$ is the mod $p$ Moore spectrum.

Determination of the homotopy groups of the $L_n$-localized sphere spectrum is one of the key problems to understanding the category of $L_n$-localized spectra. It is well known that $\pi_*(L_0S^0) = \mathbf{Q}$. The homotopy groups $\pi_*(L_1S^0)$ are determined by Ravenel [19] and $\pi_*(L_2S^0)$ is determined for $p > 3$ by Yabe and the author [24]. Thus the next case will be $\pi_*(L_2S^0)$ at the prime 3. At the prime $p > 3$, the homotopy groups $\pi_*(L_2S^0)$ are obtained from $\pi_*(L_2V(1))$ by the $v_1$- and $p$-Bockstein spectral sequences. Furthermore, at $p > 3$, the homotopy groups $\pi_*(L_2S^0)$ are isomorphic to the $E_2$-term of the Adams-Novikov spectral sequence, and they can be determined purely algebraically. Besides, $\pi_*(L_2V(1))$ is obtained from the cohomology of the Morava stabilizer algebra $S_2$, which is computed by Ravenel (*cf.* [20]). Different from the case $p > 3$, the homotopy groups at the prime 3 are not isomorphic to the $E_2$-term of the Adams-Novikov spectral sequence, and besides the $E_2$-term of the Adams-Novikov spectral sequence for computing $\pi_*(L_2V(1))$ is expressed by the language of cohomology of groups [4] (*cf.* [3], [28]). Actually, Henn noticed a mistake in [20, Th. 6.3.23] and gave the correction of it in [4] from the viewpoint of cohomology of groups. In this paper, we provide another verification of this result using the cobar complex in Section 5.

---

Received by the editors October 31, 1995.
1991 *Mathematics Subject Classification.* Primary 55Q45, 55Q10, 55Q52.

The homotopy groups $\pi_*(L_nV(n-1))$ are computed by Ravenel [20] for $n \le 3$ and $p > n+1$, in which case the Adams-Novikov spectral sequence collapses. So the next case in this sense is $p = n+1$, whose case involves non-trivial Adams-Novikov differentials. When $n = 1$, this is a celebrated result of Mahowald (*cf.* [6],[7],[8]). We study the case $n = 2$ here in this point of view.

In order to state our results, we introduce the notations:

$$K(2)_* = \mathbf{Z}/3[v_2, v_2^{-1}], \quad K = \mathbf{Z}/3[v_2^9, v_2^{-9}],$$
$$V = \mathbf{Z}/3\{v_2^j \mid j = 0,1,5\}, \quad \overline{V} = \mathbf{Z}/3\{v_2^j \mid j = 2,3,4,6,7,8\},$$
$$P = \mathbf{Z}/3[b_{10}] \quad \text{and} \quad P_n = P/(b_{10}^n).$$

Our main result is the following shown in Section 10.

**Theorem A.** *The homotopy groups $\pi_*(L_2V(1))$ at the prime 3 are isomorphic to the tensor product of the exterior algebra $\Lambda(\zeta_2)$ and the direct sum of $K$-modules*

$$P_2 \otimes \overline{V} \otimes K\{a_{-21}, a'_{-24}\} \oplus P_3 \otimes \overline{V} \otimes K\{a_{51}, a'_{38}\}$$
$$\oplus P_4 \otimes V \otimes K\{a_{82}, a'_{-3})\} \oplus P_5 \otimes V \otimes K\{a_0, a'_{69}\}$$

*for the generators $\zeta_2 \in \pi_{-1}(L_2V(1))$, $a_l \in \pi_l(L_2V(1))$ and $a'_l \in \pi_{48k+l}(L_2V(1))$. Here, $k$ is a fixed integer of $\{0,1,2\}$.*

Unfortunately, the integer $k \in \{0,1,2\}$ stays undetermined but this theorem shows the size of the homotopy groups. The homotopy groups are computed by the Adams-Novikov spectral sequence, and are just the $E_{10}$-term (see Theorem 10.6) for them, in fact, $E_{10}^s = 0$ if $s > 12$.

On the way computing it, we determine the homotopy groups $\pi_*(L_2X \wedge V(1))$ in Section 4 for the 8-skeleton $X$ of $BP$ with $BP_*(X) = BP_*\{1, a, b\}$, where $|a| = 4$ and $|b| = 8$.

**Theorem B.** *The homotopy groups $\pi_*(L_2X \wedge V(1))$ are isomorphic to the tensor product of the exterior algebra $\Lambda(\zeta_2)$ and the $K(2)_*$-module generated by $b_0$, $b_5$, $b_6$, $b_{10}$, $b_{11}$ and $b_{15}$ for the generators $b_l \in \pi_l(L_2X \wedge V(1))$.*

This is used to determine the $E_2$-term of the Adams-Novikov spectral sequence for computing $\pi_*(L_2V(1))$ (see Theorem 5.8). To show the relation $d_r(x\zeta_2) \in \zeta_2 E_r^*(L_2V(1))$ of the differential of the Adams-Novikov spectral sequence $\{E_r^*(L_2V(1))\}$ for $L_2V(1)$ (see Lemma 6.9), we compute the homotopy groups of $L_2V_1$ for

$$V_i = V(1) \cup_{\beta_1^i} \Sigma^{10i+1}V(1)$$

in Section 6:

**Theorem C.** *The homotopy groups $\pi_*(L_2V_1)$ are isomorphic to the tensor product of the exterior algebra $\Lambda(\zeta_2)$ and the $K(2)_*$-module generated by $a_0$, $a_3$, $a_6$, $a_{11}$, $a_{13}$, $a_{21}$, $a_{34}$ and $a_{40}$ for the generators $a_l \in \pi_l(L_2V_1)$.*

In the same way, we determine $\pi_*(L_2V_2)$. Furthermore, we compute $\pi_*(L_2V_3)$ to determine the Adams-Novikov differential $d_5$ on $L_2V(1)$ in Section 9.

**Theorem D.** *The homotopy groups $\pi_*(L_2V_2)$ and $\pi_*(L_2V_3)$ are given as follows:*

$$\pi_*(L_2V_2) = \pi_*(L_2V_1) \otimes \Lambda(\beta_1), \quad \textit{and}$$
$$\pi_*(L_2V_3) = \Lambda(\zeta_2) \otimes K \otimes (P_3 \otimes V \otimes F_1 \oplus \Lambda(g_3) \otimes \overline{V} \otimes (P_2 \otimes F_2 \oplus P_3 \otimes F_3)).$$

*Here $xg_3 \in \pi_*(L_2V_3)$ denotes an element such that $j_*(xg_3) = x$ for the projection $j : V_3 \to \Sigma^{31}V(1)$, and*

$$F_1 = \mathbf{Z}/3\{a_0, a_{51}, a_{-21}, a_{82}, a'_{-3}, a'_{69}, a'_{38}, a'_{-24}\},$$
$$F_2 = \mathbf{Z}/3\{a_{-21}, a'_{-24}\} \quad \textit{and} \quad F_3 = \mathbf{Z}/3\{a_{51}, a'_{38}\},$$

*for the elements $a_l$ and $a'_l$ in Theorem A.*

Pemmaraju [16] recently shows that there exists a self map $B : \Sigma^{144}V(1) \to V(1)$ with $BP_*(B) = v_2^9$. But here we show that $\pi_*(L_2V(1))$ is a $\mathbf{Z}/3[v_2^9, v_2^{-9}]$-module in a different way from his, using the homotopy element $\beta_{6/3} \in \pi_*(S^0)$. Furthermore, his result shows the existence of $\beta$-elements $\beta_t$ with $t \equiv 0, 1, 2, 3, 5, 6$ (9) (see Section 2 for the definition of the $\beta$-elements). As a corollary of Theorem A, we have

**Theorem E.** *The $\beta$-element $\beta_t$ exists in $\pi_*(L_2S^0)$ if $t \equiv 0, 1, 5$ (9).*

On the non-existence, we obtain, in the last section, the following

**Theorem F.** *In the homotopy groups $\pi_*(S^0)$, the $\beta$-element $\beta_t$ does not exist if $t \equiv 4, 7, 8$ (9).*

Note that for the case $p > 3$, the $\beta$-element $\beta_t$ exists in $\pi_*(S^0)$ for any $t > 0$. More generally, the $\beta$-elements $\beta_{pt/p}$ and $\beta_{p^2t/p,2}$ are also shown to exist for $p > 3$ by Oka (*cf.* [20]). See Section 10 for the definition of these $\beta$-elements. At the prime 3, we show a generalization of Ravenel's odd primary Kervaire invariant theorem:

**Theorem G.** *The $\beta$-elements $\beta_{9t+3/3}$, $\beta_{9s/3,2}$ and $\beta_{3^is/3^i}$ do not exist in the homotopy groups $\pi_*(S^0)$ for $t \geq 1$, $s \not\equiv 0$ (3) and $i > 1$.*

Recall the theorem of Hopkins-Gross that the Brown-Comenetz dual $I_nF$ of a finite spectrum $F$ with $K(n)_*(F) \neq 0$ and $K(n-1)_*(F) = 0$ is homotopic to the Spanier-Whitehead dual $L_nDF$ of $F$ up to suspension if $p1_F \sim 0$ and $p$ is large enough to satisfy the inequality $2p - 2 \geq \max\{n^2, 2n+2\}$. Our case $n = 2$ and $p = 3$ does not satisfy the condition, but it seems to hold true for our case. Here we can prove it for $F = V_1$ in Section 6 by using Theorem C.

**Theorem H.** $\Sigma^{-39}L_2V_1 \simeq I_2V_1$.

As is seen in the chart of Theorem 9.1, this kind of duality seems to hold even at $p = 3$. If we know the duality of this kind for $L_2V(1)$ previously, the computation will be much easier.

This paper is organized as follows:

1. Some properties of the Adams-Novikov spectral sequence.
2. Some homotopy elements in $\pi_*(V(1))$
3. Ravenel's spectral sequence
4. The homotopy groups $\pi_*(L_2X \wedge V(1))$
5. The Adams-Novikov $E_2$-term for $L_2V(1)$
6. The homotopy groups of $L_2V_1$
7. $\mathbf{Z}/3[v_2^9, v_2^{-9}]$-module structure
8. The Adams-Novikov differential on $L_2V(1)$
9. The homotopy groups of $L_2V_2$ and $L_2V_3$
10. The $E_{10}$-term of the Adams-Novikov spectral sequence for $L_2V$
11. The non-existence of $\beta$-elements

The author would like to thank Don Davis for carefully reading the first draft of this paper, and not only kindly pointing out gaps in the proof of a lemma which is here Lemma 9.7, but also suggesting the author to prove that $v_2^5$ detects a homotopy element which first was quoted from Pemmaraju's result.

## 1. Some properties of the Adams-Novikov spectral sequence

Throughout this paper, we consider everything localized at the prime 3. By $BP$, we denote the Brown-Peterson spectrum whose coefficient ring and self homology are polynomial algebras

$$BP_* = \mathbf{Z}_{(3)}[v_1, v_2, \cdots] \quad \text{and} \quad BP_*(BP) = BP_*[t_1, t_2, \cdots]$$

over the generators $v_i$ and $t_i$ with degree $|v_i| = 2(3^i - 1) = |t_i|$ for $i > 0$. This gives rise to a Hopf algebroid (*cf.* [1]). Consider the $BP_*(BP)$-comodule algebra $E(2)_* = \mathbf{Z}_{(3)}[v_1, v_2, v_2^{-1}]$ whose $BP_*$-action is given by sending $v_i$ to $v_i$ for $i \le 2$ and to 0 otherwise. Then $E(2)_*(-) = E(2)_* \otimes_{BP_*} BP_*(-)$ is a homology theory by the Landweber exact functor theorem, and we denote $E(2)$ for the spectrum representing the theory. It is a ring spectrum and yields the Hopf algebroid $(A, \Gamma) = (E(2)_*, E(2)_*(E(2)))$ with

$$\Gamma = E(2)_*(E(2)) = E(2)_*[t_1, t_2, \cdots]/(u_i : i > 2). \tag{1.1}$$

Here $u_i$ denotes the image of $v_i$ under the homomorphism $BP_* \xrightarrow{\eta_R} BP_*(BP) \to E(2)_*[t_1, t_2, \cdots]$ for the right unit $\eta_R$ of the Hopf algebroid $BP_*(BP)$. Since $\Gamma$ is flat over $A$, the category of $\Gamma$-comodules has enough injectives, and $\mathrm{Ext}_\Gamma(A, -)$ is defined to be a derived functor of $\mathrm{Hom}_\Gamma(A, -)$.

Recall [2] the construction of the Adams-Novikov spectral sequence. Let $E$ be a ring spectrum with the unit map $i : S^0 \to E$, and $\overline{E}$ the cofiber of $i$. Then we have the exact couple

$$\begin{tikzcd}
\overline{E}^s \wedge G \arrow[rr, dash, "k^{s+1}"] \arrow[dr, dash, "i\wedge 1"'] & & \overline{E}^{s+1} \wedge G \\
& E \wedge \overline{E}^s \wedge G \arrow[ur, dash, "j\wedge 1"'] &
\end{tikzcd}$$

for a spectrum $G$. Consider the cofiber $\overline{E}_s$ of $k^s : \overline{E}^s \to \Sigma^s S^0$, and we have another exact couple

$$\begin{tikzcd}
\overline{E}_s \wedge G \arrow[rr, dash, "k_{s+1}"] \arrow[dr, dash, "i_s"'] & & \overline{E}_{s+1} \wedge G \\
& E \wedge \overline{E}^s \wedge G. \arrow[ur, dash, "j_s"'] &
\end{tikzcd}$$

Applying the homology functor $\pi_*(-)$ to this exact couple, we obtain the spectral sequence abutting to $\pi_*(G)$. This spectral sequence is called a *generalized Adams spectral sequence* based on $E$. This exact couple also gives rise to a finite spectral sequence converging to the homotopy groups $\pi_*(\overline{E}_n \wedge G)$ with the same $E_1$-term as the generalized Adams spectral sequence. Let $H$ be a spectrum, and $E_r^*(H)$ denote the $E_r$-term of these spectral sequences abutting or converging to $\pi_*(H)$. The following is an easy consequence of the definition.

**Lemma 1.2.** *For any integers $n, r > 0$, $E_r^s(\overline{E}_n \wedge G) = 0$ if $s \geq n$, and $E_r^s(G \wedge \overline{E}_n) = E_r^s(G)$ if $s \leq n - r$.*

Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

be a cofiber sequence with $E_*(h) = 0$. Then we have short exact sequences of $E_1$-terms which yield a long exact sequence

$$\cdots \longrightarrow E_2^s(X) \xrightarrow{f_*} E_2^s(Y) \xrightarrow{g_*} E_2^s(Z) \xrightarrow{\partial} E_2^{s+1}(X) \longrightarrow \cdots \tag{1.3}$$

of $E_2$-terms, where $\partial$ denotes the connecting homomorphism. In the same way as [10, Th. 4.1], we see the following

**Lemma 1.4.** *Suppose that the differential $d_r : E_r^s(W) \to E_r^{s+r}(W)$ is trivial if $r < n$ for $W = X, Y, Z$. If $g_*(y)$ is a non-trivial permanent cycle for $y \in E_n^s(Y) = E_2(Y)$, then*

$$d_n(y) = f_*(\overline{h \circ z}) \in E_n^{s+n}(Z) = E_2^{s+n}(Z),$$

*where $z$ denotes a homotopy element representing $g_*(y)$, and $\overline{x}$ denotes the projection of a homotopy element $x$ of $\pi_*(X)$ into $E_2^{s+n}(X)$.*

Take now $E = E(2)$ (resp. $= BP$). Then the spectral sequence converges to $\pi_*(L_2G)$ (resp. $\pi_*(G)$) if $G$ is connective (*cf.* [19]), and we call it the *Adams-Novikov spectral sequence*. Here $L_2$ denotes the Bousfield localization functor with respect to $E(2)$. Let $E_r^{s,t}(L_2G)$ (resp. $E_r^{s,t}(G)$) denote the $E_r$-term of the Adams-Novikov spectral sequence and it is known that

$$\begin{array}{l} E_2^{s,t}(L_2G) = \mathrm{Ext}_\Gamma^{s,t}(A, E(2)_*(G)) \\ (\text{resp. } E_2^{s,t}(G) = \mathrm{Ext}_{BP_*(BP)}^{s,t}(BP_*, BP_*(G))). \end{array}$$

Now recall the construction of the Toda-Smith spectrum $V(1)$. Let $M$ denote the cofiber of $3 \in \pi_0(S^0) = \mathbf{Z}$ and we have a cofiber sequence

$$S^0 \xrightarrow{3} S^0 \xrightarrow{i} M \xrightarrow{j} S^1. \tag{1.5}$$

Now the *Toda-Smith spectrum* $V(1)$ is defined to be the cofiber of the Adams map $\alpha \in [M, M]_4$ which is characterized by $BP_*(\alpha) = v_1$. Thus we have the cofiber sequence

$$\Sigma^4 M \xrightarrow{\alpha} M \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^5 M. \tag{1.6}$$

Since $V(1)$ is an $S^0$- and $M$-module spectrum (*cf.* [27]), it is well known that

$$\begin{array}{l} \textit{For } x \in E_r^{s,t}(S) \textit{ and } y \in E_r^{s',t'}(L_2V(1)), \\ d_r(xy) = d_r(x)y + (-1)^{t-s} x d_r(y) \ (\textit{cf.}\ \ [20, \text{Th. } 2.3.3]), \end{array} \tag{1.7}$$

*where* $S = S^0$ *or* $M$.

The Bockstein operation on $V(1)$ is defined by the composition $i_1 j_1$ of maps in (1.6), which induces the usual Bockstein operation $\delta$ on $E_2^*(L_2V(1))$ given by

$$\delta = i_{1*}\partial_1 \tag{1.8}$$

for the connecting homomorphism $\partial_1 : E_2^*(L_2V(1)) \to E_2^{*+1}(L_2M)$.

**Lemma 1.9.** *Let $F$ denote a finite spectrum such that $E(2)_*(F)$ is free over $E(2)_*$, and $x$ and $y$, elements of $E_2^*(L_2V(1)\wedge F)$. Suppose that $x$ and $y$ are permanent cycles such that $d_r(xy)=0$ for $r<5$, and that $\partial(xy)\neq 0$. Then*

$$d_5(xy)=i_{1*}(\omega(x))\delta(y)\in E_2^*(L_2V(1)\wedge F),$$

*where $\omega(x)\in E_2^*(L_2M\wedge F)$ denotes the projection of the Toda bracket $\langle\alpha, j_{1*}(\tilde{x}),\alpha\rangle$ for the homotopy element $\tilde{x}$ detected by $x$.*

*Proof.* The proof is almost identical to that of [10, Th. 4.2]. Notice that $\partial(xy)$ equals the Massey product $\langle\partial(x), v_1, \partial(y)\rangle$, and that $v_1$ detects the homotopy element $\alpha$. We denote a homotopy element $j_1\tilde{x}$ detected by $\partial(x)$. Then $\partial(xy)$ detects a homotopy element given by the Toda bracket $\langle j_1\tilde{x},\alpha, j_1\tilde{y}\rangle$. By Lemma 1.4, we compute

$$\begin{aligned} d_5(xy) &= i_{1*}(\overline{\alpha\langle j_1\tilde{x},\alpha,j_1\tilde{y}\rangle}) \\ &= i_{1*}(\overline{\langle\alpha, j_1\tilde{x},\alpha\rangle j_1\tilde{y}}) \\ &= i_*(\omega)\delta(y). \quad \text{q.e.d.} \end{aligned}$$

## 2. Some homotopy elements in $\pi_*(V(1))$

Throughout this section we consider the Adams-Novikov spectral sequence $\{E_r^*(X)\}$ for computing $\pi_*(X)$. First we review the definition of $\alpha$- and $\beta$-elements. Let $\partial: E_2^s(M)\to E_2^{s+1}(S^0)$ and $\partial_1: E_2^s(V(1))\to E_2^{s+1}(M)$ be the connecting homomorphisms given in (1.3) from the cofiber sequences (1.5) and (1.6), respectively. Recall [20, Cor. 4.3.21] Landweber's formula $\eta_R(v_n)\equiv v_n+v_{n-1}t_1^{3^{n-1}}-v_{n-1}^3t_1$ mod $(3,v_1,\cdots,v_{n-2})$. Then, we see that $v_1^t\in E_2^0(M)$ and $v_2^t\in E_2^0(V(1))$ for $t>0$. Now, $\alpha_t$, $\beta_t'$ and $\beta_t$ are defined by

$$\begin{aligned} &\alpha_t=\partial(v_1^t)\in E_2^1(S^0),\\ &\beta_t'=\partial_1(v_2^t)\in E_2^1(M) \quad \text{and}\\ &\beta_t=\partial\partial_1(v_2^t)\in E_2^2(S^0). \end{aligned}\tag{2.1}$$

In [20], we find that $\alpha_t$ for $t>0$ and $\beta_t$ for $t=1,2,3,5,6$ are permanent as well as $\beta_t'$ for $t=1,2,3,5,6$. Send these elements by the induced maps from the inclusions $i_1i: S^0\to V(1)$ and $i_1: M\to V(1)$, and we obtain the same named elements in $E_2^s(V(1))$. These are represented by elements of the cobar complex as follows (*cf.* [14, Lemma 5.4], [20]):

$$\begin{aligned} &\alpha_1=h_{10}, \quad \beta_t'=tv_2^{t-1}h_{11},\\ &\beta_{3k+1}=v_2^{3k}b_{10} \quad \text{and} \quad \beta_{3k+2}=v_2^{3k+1}h_{11}\zeta_2, \end{aligned}\tag{2.2}$$

for non-negative integers $k,t$ with $t>0$. In a similar way, we also have another $\beta$-element $\beta_{3t/3}$, which is represented as follows (*cf.* [14, Lemma 5.4]):

$$\beta_{3s/3}=sv_2^{3s-3}b_{11} \quad \text{and} \quad \beta_{9s/3,2}=sv_2^{9s-3}b_{11}.\tag{2.3}$$

Here the homology classes $h_{1i}$, $b_{1i}$ and $\zeta_2$ are represented by cocycle of the cobar complex $\Omega_\Gamma^*E(2)_*$ as follows:

$$h_{1i}=[t_1^{3^i}],\quad b_{1i}=[(t_1\otimes t_1^2+t_1^2\otimes t_1)^{3^i}] \quad \text{and} \quad \zeta_2=[v_2^{-1}t_2+v_2^{-3}(t_2^3-t_1^{12})].$$

For the Adams-Novikov spectral sequence for $\pi_*(L_2V(1))$, we have the following

**Lemma 2.4.** *In the $E_2$-term $E_2^*(L_2V(1))$ of the Adams-Novikov spectral sequence,*

$$h_{10},\quad v_2^5h_{10},\quad h_{11},\quad v_2h_{11},\quad v_2^4h_{11},$$
$$b_{10},\quad v_2h_{11}\zeta_2 \quad \textit{and} \quad v_2^3b_{11}$$

*are all permanent cycles. Therefore, if $x$ denotes one of the elements, then*

$$d_r(yx) = d_r(y)x$$

*for any $y \in E_r^*(L_2V(1))$.*

*Proof.* By [20, Table A3.4], we see that $\alpha_1$, $\beta_1$, $\beta_2$, $\beta_5$ and $\beta_{6/3}$ are essential homotopy elements of $\pi_*(S^0)$, and these are pulled back to $\pi_*(M)$ and denoted by $\alpha_1'$, $\beta_1'$, $\beta_2'$ and $\beta_{6/3}'$, since they have order 3. By (2.2) and (2.3), we see that these homotopy elements represent the elements of the $E_2$-term for $V(1)$ as follows:

$$\alpha_1 = h_{10},\quad \beta_1 = b_{10},\quad \beta_2 = -v_2h_{11}\zeta_2,\quad \beta_5 = -v_2^4h_{11}\zeta_2,\quad \beta_{6/3} = -v_2^3b_{11},$$
$$\alpha_1' = v_1,\quad \beta_1' = h_{11},\quad \beta_2' = -v_2h_{11},\quad \beta_5' = -v_2^4h_{11},\quad \text{and}\quad \beta_{6/3}' = -v_2^5h_{10}.$$

Here $\beta_{6/3}'$ is also read off from [14, (4.5)] using the relation $t_1^9 \equiv v_2^2t_1 \bmod (3, v_1)$ obtained from $u_3 = 0$ in (1.1). This implies the former half. The latter half follows from (1.7), immediately. q.e.d.

From here on, we consider the homotopy groups of $V(1)$, the unlocalized one. If $v_2^t$ of the $E_2$-term survives to $\beta^{(t)}$ of $\pi_*(V(1))$, then $\beta_t$ in the $E_2$-term detects the same named element

$$\beta_t = jj_1(\beta^{(t)}) \in \pi_{16t-6}(S^0), \tag{2.5}$$

for $j$ and $j_1$ in (1.5) and (1.6) by the Geometric Boundary theorem (*cf.* [20, Th. 2.3.4]. As to $\beta^{(t)}$, we have the following

**Theorem 2.6.** *If $t = 0, 1$ or $5$, then $v_2^t$ survives to $E_\infty$-term, that is, it detects a homotopy element $\beta^{(t)} \in \pi_*(V(1))$.*

*Proof.* For $t = 0$, we put $\beta^{(0)} = 1 \in \pi_0(V(1))$. Since $E_2^{s,t}(V(1)) = 0$ unless $t \equiv 0 \bmod 4$ and $E_2^{s+2,s+17} = 0$ if $s > 1$, we see that $d_r(v_2) = 0$ in $E_r^r(V(1))$, and that $v_2$ detects the homotopy element $\beta^{(1)}$.

Now we consider $v_2^5$. We will show that the homotopy element $\beta_5 \in \pi_{74}(S^0)$ is pulled back to $\pi_{80}(V(1))$ by the map $jj_1 : \Sigma^6V(1) \to S^0$. If this is shown, then we take $\beta^{(5)} = (jj_1)_*^{-1}(\beta_5)$ by virtue of (2.5). The table of Ravenel's book [20, Table A.3.4] shows

$$\pi_{75}(S^0) = \mathbf{Z}/3\{\alpha_{19}\} \oplus \mathbf{Z}/9\{x_{75}\} \quad \text{and} \quad \pi_{74}(S^0) = \mathbf{Z}/3\{\beta_5\}; \quad \text{and}$$
$$\pi_{79}(S^0) = \mathbf{Z}/3\{\alpha_{20}\} \quad \text{and} \quad \pi_{78}(S^0) = \mathbf{Z}/3\{\beta_1x_{68}\}.$$

Consider the long exact sequence

$$\cdots \xrightarrow{3} \pi_{75}(S^0) \xrightarrow{i_*} \pi_{75}(M) \xrightarrow{j_*} \pi_{74}(S^0) \xrightarrow{3} \cdots$$

associated to the cofiber sequence (1.5), and we obtain

$$\pi_{75}(M) = \mathbf{Z}/3\{i_*(\alpha_{19}), i_*(x_{75}), \widetilde{\beta}_5\}$$

for $\widetilde{\beta}_5$ such that $j_*(\widetilde{\beta}_5) = \beta_5$. In the same way, we obtain

$$\pi_{79}(M) = \mathbf{Z}/3\{i_*(\alpha_{20}), \widetilde{\beta}_1x_{68}\}$$

for $\widetilde{\beta}_1$ such that $j_*(\widetilde{\beta}_1) = \beta_1$. Note that $\alpha i x_{75} \in \pi_{79}(M)$ by degree reason, and we compute

$$j_*(\alpha i x_{75}) = \alpha_1 x_{75} = \alpha_1 \langle \alpha_1, \alpha_1, x_{68} \rangle = \langle \alpha_1, \alpha_1, \alpha_1 \rangle x_{68} = \beta_1 x_{68} = j_*(\widetilde{\beta}_1 x_{68}).$$

Therefore, we put

$$\alpha i x_{75} = \widetilde{\beta}_1 x_{68} + k i_*(\alpha_{20})$$

for some $k \in \mathbf{Z}/3$. Now consider the Adams-Novikov filtration, which is defined as follows:

$$\text{(2.7)} \qquad \begin{array}{l} \text{filt } x = n \text{ for } x \in \pi_*(X) \text{ if and only if } x \text{ is detected by} \\ \text{a non-trivial element of } E_2^n(X). \end{array}$$

Since $i_*(\alpha_{20}) = \alpha_{20}$ in the $E_2$-term, we see that filt $i_*(\alpha_{20}) = 1$. We also read off filt $\alpha i x_{75} =$ filt $\widetilde{\beta}_1 x_{68} = 5$ from [20]. Therefore, $k = 0$ and so

$$\text{(2.8)} \qquad \alpha i x_{75} = \widetilde{\beta}_1 x_{68}.$$

We note that $\widetilde{\beta}_5$ is detected by the element $\partial_1(v_2^5)$ in the $E_2$-term. We also see that $v_1 \partial_1(v_2^5) = 0$ in the $E_2$-term by the definition of $\partial_1$, which means filt $\alpha\widetilde{\beta_5} > 1$. Furthermore, filt $i_*(\alpha_{20}) = 1 =$ filt $\widetilde{\beta}_1$. Therefore we read off the relation

$$\text{(2.9)} \qquad \alpha \widetilde{\beta}_5 = k \widetilde{\beta}_1 x_{68}$$

for some $k \in \mathbf{Z}/3$, from the homotopy group $\pi_{79}(M)$. Now define

$$\beta_5' = \widetilde{\beta}_5 - k i_*(x_{75}) \in \pi_{75}(M).$$

Then, $j_*(\beta_5') = \beta_5$, and (2.8) and (2.9) show

$$\alpha \beta_5' = 0.$$

This implies the existence of an element $\beta^{(5)} \in \pi_{80}(V(1))$ such that $j_{1*}(\beta^{(5)}) = \beta_5'$, as desired. q.e.d.

## 3. Ravenel's spectral sequence

Let $X$, $Y$ and $Y'$ be the spectra defined by

(3.1)

$$X = S^0 \cup_{\alpha_1} e^4 \cup_{-\alpha_1} e^8, \quad Y = S^0 \cup_{\alpha_1} e^4 \quad \text{and} \quad Y' = S^0 \cup_{-\alpha_1} e^4.$$

Note that $X$ and $Y$ are the 8-skeleton and the 4-skeleton of $BP$, respectively. Then we have cofiber sequences

$$Y' \xrightarrow{i} X \xrightarrow{j} S^8 \xrightarrow{k} \Sigma Y' \quad \text{and} \quad S^0 \xrightarrow{i'} X \xrightarrow{j'} \Sigma^4 Y' \xrightarrow{k'} S^1.$$

Here $i : Y' \to X$ is defined to be the composition $Y' \xrightarrow{i} Y \subset X$ for $i$ fitting into the commutative diagram

(3.2)

$$\begin{tikzcd} S^3 \arrow[r, dash, "-\alpha_1"] \arrow[d, dash, "-1"'] & S^0 \arrow[r, dash] \arrow[d, dash] & Y' \arrow[d, dash, "i"] \\ S^3 \arrow[r, dash, "\alpha_1"'] & S^0 \arrow[r, dash] & Y. \end{tikzcd}$$

Applying the homotopy functor $\pi_*(-)$, we have two long exact sequences

$$\cdots \longrightarrow \pi_*(G\wedge Y') \xrightarrow{i_*} \pi_*(G\wedge X) \xrightarrow{j_*} \pi_{*-8}(G) \longrightarrow \cdots, \quad \text{and}$$
$$\cdots \longrightarrow \pi_*(G) \xrightarrow{i'_*} \pi_*(G\wedge X) \xrightarrow{j'_*} \pi_{*-4}(G\wedge Y') \longrightarrow \cdots$$

for any spectrum $G$. These associate the spectral sequence

(3.3) $$E_1^{s,t} = (\Lambda(\alpha_1)\otimes \mathbf{Z}_{(3)}[\beta_1]\otimes \pi_*(G\wedge X))^{s,t} \Longrightarrow \pi_{t-s}(G)$$

with bidegree $||\alpha_1|| = (1,4)$, $||\beta_1|| = (2,12)$ and $||x|| = (0,t)$ for $x\in \pi_t(G\wedge X)$.

Taking $E(2)\wedge G$ for $G$ above, we have the short exact sequences

(3.4)
$$0 \longrightarrow E(2)_*(G\wedge Y') \xrightarrow{i_*} E(2)_*(G\wedge X) \xrightarrow{j_*} E(2)_{*-8}(G) \longrightarrow 0, \quad \text{and}$$
$$0 \longrightarrow E(2)_*(G) \xrightarrow{i'_*} E(2)_*(G\wedge X) \xrightarrow{j'_*} E(2)_{*-4}(G\wedge Y') \longrightarrow 0,$$

since $E(2)_*(X)$ and $E(2)_*(Y')$ are $E(2)_*$-free over generators with degree 0 mod 4. Applying $\mathrm{Ext}_\Gamma^{*,*}(A,-)$ to these, we obtain an exact couple which gives us a spectral sequence

(3.5) $$\begin{aligned} E_1^{s,t} = (\Lambda(h_{10})\otimes \mathbf{Z}_{(3)}[b_{10}]\otimes \mathrm{Ext}_\Gamma^{*,*}(A, E(2)_*(G\wedge X)))^{s,t} \\ \Longrightarrow \mathrm{Ext}_\Gamma^{s,t}(A, E(2)_*(G)). \end{aligned}$$

Here the bidegrees are given as follows:

$$\begin{aligned} &||h_{10}|| = (1,4), \quad ||b_{10}|| = (2,12) \quad \text{and} \\ &||x|| = (s,t) \quad \text{for} \quad x\in \mathrm{Ext}_\Gamma^{s,t}(A, E(2)_*(G\wedge X)). \end{aligned}$$

Note that the classes $h_{10}$ and $b_{10}$ converge to the generators $\alpha_1$ and $\beta_1$, respectively, of the $E_2$-term of the Adams-Novikov spectral sequence for spheres (*cf.* (2.2)).

## 4. The homotopy groups $\pi_*(L_2X\wedge V(1))$

Recall (1.6) the definition of the Toda-Smith spectrum $V(1)$. We use the following notation as is done in [15]:

$$V = V(1) \quad \text{and} \quad VX = V\wedge X = V(1)\wedge X,$$

and obtain that

$$E(2)_*(V) = K(2)_* = \mathbf{Z}/3[v_2, v_2^{-1}] \quad \text{and} \quad E(2)_*(VX) = K(2)_*[t_1]/(t_1^3),$$

as a $\Gamma$-comodule. Then the $E_2$-term of the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(L_2VX)$ is

$$\mathrm{Ext}_\Gamma^{s,t}(A, K(2)_*[t_1]/(t_1^3)).$$

In order to compute this, we use a change of rings theorem. For this purpose, we introduce Hopf algebras over $\mathbf{Z}/3$

$$\begin{gathered} S_* = \mathbf{Z}/3[t_1]/(t_1 - t_1^9), \quad T_* = \mathbf{Z}/3[t_1^3]/(t_1^9), \\ S(2)_* = \mathbf{Z}/3[t_1, t_2, \cdots]/(t_i - t_i^9 : i\ge 1) \quad \text{and} \\ S(2,2)_* = \mathbf{Z}/3[t_2, t_3, \cdots]/(t_i - t_i^9 : i > 1), \end{gathered}$$

whose structure maps are read off from that of $\Gamma$, or that of $BP_*(BP)$ (*cf.* [20, Chap. 4]). For example, for $\Delta : S(2)_* \to S(2)_*\otimes S(2)_*$,

(4.1) $$\begin{aligned} \Delta(t_1) &= t_1\otimes 1 + 1\otimes t_1, \\ \Delta(t_2) &= t_2\otimes 1 + t_1\otimes t_1^3 + 1\otimes t_2 \quad \text{and} \\ \Delta(t_3) &= t_3\otimes 1 + t_1\otimes t_2^3 + t_2\otimes t_1^9 + 1\otimes t_3 + b_{11}. \end{aligned}$$

Here,

$$b_{11} = t_1^3 \otimes t_1^6 + t_1^6 \otimes t_1^3.$$

We then have the extension of Hopf algebras over $\mathbf{Z}/3$

$$S_* \longrightarrow S(2)_* \longrightarrow S(2,2),$$

by which we have the Cartan-Eilenberg spectral sequence (*cf.* [20, A1.3.14])

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}^s_{S_*}(\mathbf{Z}/3, \mathrm{Ext}^t_{S(2,2)_*}(\mathbf{Z}/3, \mathbf{Z}/3[t_1]/(t_1^3))) \\ &\Longrightarrow \mathrm{Ext}^{s+t}_{S(2)_*}(\mathbf{Z}/3, \mathbf{Z}/3[t_1]/(t_1^3)). \end{aligned}$$

Here the $E_2$-term is computed as follows:

$$\begin{aligned} E_2^{s,t} &= \mathrm{Ext}^s_{S_*}(\mathbf{Z}/3, \mathrm{Ext}^t_{S(2,2)_*}(\mathbf{Z}/3, \mathbf{Z}/3) \otimes \mathbf{Z}/3[t_1]/(t_1^3)) \\ &= \mathrm{Ext}^s_{T_*}(\mathbf{Z}/3, \mathrm{Ext}^t_{S(2,2)_*}(\mathbf{Z}/3, \mathbf{Z}/3)), \end{aligned} \tag{4.2}$$

by the change of rings theorem [20, A1.3.12], since $\mathbf{Z}/3[t_1]/(t_1^3) = S_* \square_{T_*} \mathbf{Z}/3$ seen by the structure (4.1).

**Lemma 4.3.** ([20, Th. 6.3.7])

$$\mathrm{Ext}^*_{S(2,2)}(\mathbf{Z}/3, \mathbf{Z}/3) = \Lambda(h_{20}, h_{21}, h_{30}, h_{31}).$$

*Here $h_{ij}$ is represented by $t_i^{3j}$ of the cobar complex $\Omega^1_{S(2,2)_*}\mathbf{Z}/3 = S(2,2)_*$.*

**Theorem 4.4.** *The $E_2$-term of the Adams-Novikov spectral sequence for $\pi_*(L_2VX)$ is given by*

$$E_2^{*,*}(L_2VX) = \Lambda(\zeta_2) \otimes K(2)_*\{1, h_{11}, h_{20}, \xi, \varphi, h_{20}\xi\}.$$

*Here $\zeta_2$, $\xi$ and $\varphi$ are represented by $h_{20} + h_{21}$, $h_{11}h_{31}$ and $(h_{20} - h_{21})h_{31}$, and so the bidegrees of the generators are:*

$$\begin{gathered} ||\zeta_2|| = (1,0), \quad ||h_{11}|| = (1,12), \quad ||h_{20}|| = (1,0), \\ ||\xi|| = (2,8) \quad \textit{and} \quad ||\varphi|| = (2,12). \end{gathered}$$

*Proof.* By the structure map (4.1), the equation (4.2) turns into

$$E_2^{s,t} = \mathrm{Ext}^s_{T_*}(\mathbf{Z}/3, \mathbf{Z}/3) \otimes \mathrm{Ext}^t_{S(2,2)_*}(\mathbf{Z}/3, \mathbf{Z}/3),$$

and it is well known that

$$\mathrm{Ext}^s_{T_*}(\mathbf{Z}/3, \mathbf{Z}/3) = \Lambda(h_{11}) \otimes \mathbf{Z}/3[b_{11}].$$

The differential $d_2$ of the Cartan-Eilenberg spectral sequence is computed to be

$$\begin{aligned} &d_2(h_{2i}) = 0 \quad \text{for} \quad i = 0,1; \quad \text{and} \\ &d_2(h_{30}) = -b_{11} \quad \text{and} \quad d_2(h_{31}) = -h_{11}(h_{20} - h_{21}) \end{aligned}$$

by (4.1). Now note that

$$E_2^*(L_2VX) = K(2)_* \otimes \mathrm{Ext}^*_{S(2)_*}(\mathbf{Z}/3, \mathbf{Z}/3[t_1]/(t_1^3)),$$

and we obtain the theorem. q.e.d.

Since $E_2^{s,t}(L_2VX) = 0$ unless $t \equiv 0 \bmod 4$, the Adams-Novikov differential $d_r = 0$ for $r \le 4$. The theorem shows that $E_2^{s,t}(L_2VX) = 0$ for $s > 4$, which shows $d_r = 0$ for $r > 4$. Therefore, the spectral sequence collapses to the $E_2$-term. Since $V(1)$ is an $M$-module spectrum [27], there is no extension problem. Hence the $E_2$-term gives the homotopy groups of $L_2VX$.

**Corollary 4.5.** $\pi_*(L_2VX) = \Lambda(\zeta_2) \otimes K(2)_*\{1, h_{11}, h_{20}, \xi, \varphi, h_{20}\xi\}$.

## 5. The Adams-Novikov $E_2$-term for $L_2V(1)$

For the sake of convenience, we use the abbreviations:

$$\begin{aligned}
&\mathrm{Ext}^{*,*}M = \mathrm{Ext}_\Gamma^{*,*}(A, M) \quad \text{for a } \Gamma\text{-comodule } M,\\
&V = V(1), \qquad VX = V(1)\wedge X,\\
&VY = V(1)\wedge Y, \qquad VY' = V(1)\wedge Y',\\
&K(2)_* = E(2)_*(V) = \mathbf{Z}/3[v_2v_2^{-1}],\\
&KY = E(2)_*(VY) = K(2)_*\oplus K(2)_*\{a\},\\
&KY' = E(2)_*(VY') = K(2)_*\oplus K(2)_*\{a'\} \quad \text{and}\\
&KX = E(2)_*(VX) = K(2)_*\oplus K(2)_*\{a\}\oplus K(2)_*\{b\}.
\end{aligned}$$

Here $X$, $Y$ and $Y'$ are the spectra in (3.1). The $\Gamma$-comodule structure $\psi$ (resp. $\psi'$) of $KX$ and $KY$ (resp. $KY'$) satisfies

$$\psi(a) = a + t_1 \quad \text{and} \quad \psi(b) = b - at_1 + t_1^2. \quad (\text{resp. } \psi'(a') = a' - t_1).$$

**Lemma 5.1.** *The map $i : Y' \to Y$ in (3.2) induces an isomorphism $i_* : KY' \to KY$ of comodules such that $i_*(1) = 1$ and $i_*(a') = -a$.*

Taking $G$ to be $L_2V$, the short exact sequences of (3.4) induce the long ones

(5.2)

$$\begin{gathered}
\mathrm{Ext}^sKY' \xrightarrow{i_*} \mathrm{Ext}^sKX \xrightarrow{j_*} \mathrm{Ext}^sK(2)_* \xrightarrow{\partial} \mathrm{Ext}^{s+1}KY', \quad \text{and}\\
\mathrm{Ext}^sK(2)_* \xrightarrow{i'_*} \mathrm{Ext}^sKX \xrightarrow{j'_*} \mathrm{Ext}^sKY' \xrightarrow{\partial'} \mathrm{Ext}^{s+1}K(2)_*
\end{gathered}$$

with the connecting homomorphisms

$$\partial : \mathrm{Ext}^{*,*}K(2)_* \longrightarrow \mathrm{Ext}^{*+1,*}KY' \quad \text{and} \quad \partial' : \mathrm{Ext}^{*,*}KY' \longrightarrow \mathrm{Ext}^{*+1,*}K(2)_*.$$

**Lemma 5.3.** *As a $\Lambda(\zeta_2)$-module map, the maps $j_*$ and $j'_*$ of (5.2) send elements as follows:*

$$\begin{gathered}
j_*(1) = 0, \quad j_*(h_{11}) = 0, \quad j_*(h_{20}) = 0,\\
j_*(\xi) = 0, \quad j_*(\varphi) = 0, \quad j_*(h_{20}\xi) = 0; \quad \textit{and}\\
j'_*(1) = 0, \quad j'_*(h_{11}) = 0, \quad j'_*(h_{20}) = h_{11},\\
j'_*(\xi) = 0, \quad j'_*(\varphi) = \xi, \quad j'_*(h_{20}\xi) = 0.
\end{gathered}$$

*Proof.* In this proof, we omit $v_2$'s, because they just play a role adjusting the internal degrees. For a cocycle $x + ax_1 + bx_2$ of the cobar complex $\Omega_\Gamma^*KX$,

$$j_*(x + ax_1 + bx_2) = x_2 \quad \text{and} \quad j'_*(x + ax_1 + bx_2) = x_1 + ax_2,$$

where $x, x_1, x_2 \in \Omega_\Gamma^*K(2)$. We notice that $\xi$ and $\varphi$ are represented by Massey products

$$\xi = -\langle h_{11}, h_{11}, h_{20} - h_{21}\rangle \quad \text{and} \quad \varphi = \langle h_{20} - h_{21}, h_{11}, h_{20} - h_{21}\rangle.$$

Here the product of $KX$ used to define these Massey products is defined by

$$(x + ax_1 + bx_2)(y + ay_1 + by_2) = xy + a(x_1y + xy_1) + b(x_2y + x_1y_1 + xy_2).$$

Now we see easily that the cocycles $h_{11}$, $h_{20}$ and $h_{20} - h_{21}$ are represented as follows:

$$h_{11} = [t_1^3], \quad h_{20} = [t_2 + at_1^3], \quad h_{20} - h_{21} = [A + at_1^3],$$

in which $A = t_2 - t_2^3 + t_1^4$. These show the lemma for 1, $h_{11}$ and $h_{20}$.

On the other hand, $\xi$ is known to exist in $\Omega^2K(2)_*$ (*cf.* [20]), and so $j_*(\xi)$ and $j'_*(\xi)$ are both 0, and so is for $h_{20}\xi$.

Since $h_{11}(h_{20}-h_{21})=0$, there exists a cochain $z$ such that $d(z)=(A+at_1^3)\otimes t_1^3$. Using $z$, $\xi$ and $\varphi$ are represented by $\xi=[-t_1^3\otimes z+\cdots]$ and $\varphi=[-A\otimes z-at_1^3\otimes z+\cdots]$. Therefore, we obtain $j_*(\varphi)=0$ and $j'_*(\varphi)=\xi$. q.e.d.

**Lemma 5.4.** (i) *For an element $x\in \mathrm{Ext}^*K(2)_*$,*

$$\partial(x)=t_1^2\otimes x+a't_1\otimes x.$$

(ii) *For an element $y=y_1+a'y_2\in \mathrm{Ext}^*KY'$,*

$$\partial'(y)=t_1\otimes y_1+t_1^2\otimes y_2.$$

*Proof.* (i) The equation $\partial(x)=w$ follows from the relation $d(bx)=i_*(w)$ in the cobar complex $\Omega_\Gamma^*KX$ by the definition of $\partial$. Now compute

$$d(bx)=-at_1\otimes x+t_1^2\otimes x,$$

since $x$ is a cocycle.

(ii) Write $y=y_1+a'y_2$. By definition, $\partial'(y)$ follows from the computation $d(ay_1+by_2)$. Since $y$ is a cocycle, $d(y_1)=t_1\otimes y_2$ and $d(y_2)=0$. Then,

$$d(ay_1+by_2)=t_1\otimes y_1+at_1\otimes y_2-at_1\otimes y_2+t_1^2\otimes y_2.$$

q.e.d.

First we compute the $E_2$-terms of Ravenel's spectral sequence (3.5) with $G=V$:

**Lemma 5.5.** *The $E_2$-terms are isomorphic, as a $K(2)_*[b_{10}]$-module, to the $K(2)_*[b_{10}]$-module*

$$\Lambda(\zeta_2)\otimes K(2)_*[b_{10}]\{1,h_{10},h_{11},\xi,b_{11},\psi_0,\psi_1,b_{11}\xi\}.$$

*Here $\psi_0$, $\psi_1$ and $b_{11}$ are represented by $h_{10}\varphi$, $h_{20}\xi$ and $h_{10}h_{20}$, respectively, and their bidegrees are: $||\psi_0||=(3,16)$, $||\psi_1||=(3,24)$ and $||b_{11}||=(2,36)$.*

*Proof.* We have the only non-trivial differentials:

$$d_1(h_{20})=h_{10}h_{11}\quad\text{and}\quad d_1(\varphi)=h_{10}\xi, \tag{5.6}$$

which is seen by Lemmas 5.3 and 5.4. These show the lemma. q.e.d.

**Lemma 5.7.** *In the spectral sequence* (3.5) *with $G=V$,*

$$d_r=0$$

*for $r>1$.*

*Proof.* Due to [20, p.239], $h_{10}$, $h_{11}$, $b_{10}$, $b_{11}$, $\xi$ and $b_{11}\xi$ are the cocycles of $\Omega_\Gamma^*K(2)_*$. Furthermore, $\psi_k$ is represented by the Massey product $\langle h_{1k},h_{10},\xi\rangle$ for $k=0,1$, and so these are also cocycles of the complex. This means that every element in the $E_2$-term is a permanent cycle of the spectral sequence (3.5) as desired. q.e.d.

Therefore, $E_2=E_\infty$ in the spectral sequence of (3.5). So Lemma 5.5 is restated as follows:

**Theorem 5.8.** *The $E_2$-term of the Adams-Novikov spectral sequence for computing $\pi_*(L_2V(1))$ is isomorphic, as a $P$-module, to the $P$-module*

$$P\otimes_{K(2)_*}F\otimes\Lambda(\zeta_2).$$

*Here,*

$$P=K(2)_*[b_{10}]\quad\textit{and}\quad F=K(2)_*\{1,h_{10},h_{11},b_{11},\xi,\psi_0,\psi_1,b_{11}\xi\}$$

*and $\psi_k=\langle h_{1k},h_{10},\xi\rangle$ (Massey product).*

**Proposition 5.9.** *We have the following multiplicative relations:*

$$h_{10}h_{11} = 0, \quad h_{10}\xi = 0 \quad \textit{and} \quad h_{11}\xi = 0.$$

*Moreover, we have*

$$\begin{gathered} v_2^2 h_{10} b_{10} = h_{11} b_{11}, \quad v_2 h_{11} b_{10} = -h_{10} b_{11}, \\ b_{11}\xi = v_2 h_{10}\psi_1 = v_2 h_{11}\psi_0, \quad b_{10}\xi = -h_{10}\psi_0 = v_2^{-1} h_{11}\psi_1, \\ v_2^3 b_{10}^2 = -b_{11}^2, b_{10}\psi_1 = -v_2^{-1} b_{11}\psi_0 \quad \textit{and} \quad b_{10}\psi_0 = v_2^{-2} b_{11}\psi_1 \end{gathered}$$

*read off from the relation of Massey products.*

*Proof.* The triviality $h_{10}h_{11} = 0$ and $h_{10}\xi = 0$ follow from the relation (5.6). $h_{11}\xi = 0$ follows from Theorem 4.4. In fact, by degree reason, Theorem 5.8 indicates $h_{11}\xi = xv_2^{-1}b_{11}\zeta_2$ for some $x \in \mathbf{Z}/3$. The edge homomorphism sends this to the same one in $E_2^*(L_2VX)$. In $E_2^*(L_2VX)$, $h_{11}\xi = 0$, and so $x = 0$.

For the second half, we have:

$$\begin{gathered} \langle h_{10}, h_{10}, h_{10}\rangle = -b_{10}, \quad \langle h_{11}, h_{11}, h_{11}\rangle = -b_{11}, \\ \langle h_{11}, h_{10}, h_{10}\rangle = \langle h_{10}, h_{11}, h_{10}\rangle = v_2^{-1} b_{11} + v_2 h_{10}\zeta_2 \quad \text{and} \\ \langle h_{11}, h_{11}, h_{10}\rangle = v_2 b_{10} + v_2\zeta_2 h_{11}. \end{gathered}$$

Therefore, we see that

$$\begin{aligned} v_2^2 h_{10} b_{10} &= v_2\langle h_{11}, h_{11}, h_{10}\rangle h_{10} \\ &= v_2 h_{11}\langle h_{11}, h_{10}, h_{10}\rangle \\ &= v_2 h_{11}(v_2^{-1} b_{11} + h_{10}\zeta_2) \\ v_2 h_{11} b_{10} &= -v_2 h_{11}\langle h_{10}, h_{10}, h_{10}\rangle \\ &= -v_2\langle h_{11}, h_{10}, h_{10}\rangle h_{10} \\ v_2 h_{10}\psi_1 &= v_2 h_{10}\langle h_{11}, h_{10}, \xi\rangle \\ &= v_2\langle h_{10}, h_{11}, h_{10}\rangle\xi \\ v_2 h_{11}\psi_0 &= v_2 h_{11}\langle h_{10}, h_{10}, \xi\rangle \\ &= v_2\langle h_{11}, h_{10}, h_{10}\rangle\xi \end{aligned} \qquad \begin{aligned} h_{10}\psi_0 &= h_{10}\langle h_{10}, h_{10}, \xi\rangle \\ &= \langle h_{10}, h_{10}, h_{10}\rangle\xi \\ v_2^{-1} h_{11}\psi_1 &= v_2^{-1} h_{11}\langle h_{11}, h_{10}, \xi\rangle \\ &= v_2^{-1}\langle h_{11}, h_{11}, h_{10}\rangle\xi \\ b_{10}\psi_0 &= b_{10}\langle h_{10}, h_{10}, \xi\rangle \\ &= \langle b_{10}h_{10}, h_{10}, \xi\rangle \\ &= \langle v_2^{-2} b_{11}h_{11}, h_{10}, \xi\rangle \\ b_{10}\psi_1 &= b_{10}\langle h_{11}, h_{10}, \xi\rangle \\ &= \langle b_{10}h_{11}, h_{10}, \xi\rangle \\ &= \langle -v_2^{-1} b_{11}h_{10}, h_{10}, \xi\rangle. \end{aligned}$$

These have no indeterminacy by degree reason.

For the other relation,

$$v_2^3 b_{10}^2 = -v_2^3 b_{10}\langle h_{10}, h_{10}, h_{10}\rangle \subset -v_2^3\langle b_{10}h_{10}, h_{10}, h_{10}\rangle$$

which contains $-b_{11}^2$ with indeterminacy $\mathbf{Z}/3\{v_2^2 h_{10} b_{11}\zeta_2\}$. So put $v_2^3 b_{10}^2 = -b_{11}^2 + kv_2^3 h_{11} b_{10}\zeta_2$ and we will show that $k = 0$. Consider a $\Gamma$-comodule $A_i = A_*/(3, v_1^i)$ for $i > 0$. Then a short exact sequence $0 \to A_1 \to A_2 \to A_1 \to 0$, which associates a long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^s(A_1) \xrightarrow{v_1} \operatorname{Ext}^s(A_2) \xrightarrow{j} \operatorname{Ext}^s(A_1) \xrightarrow{\delta} \cdots . \tag{5.10}$$

By observing the Massey products, we see

$$v_2^3 b_{10}^2 = -b_{11}^2 + kv_2^3 h_{11} b_{10}\zeta_2 + lv_1 v_2^3 h_{10}\psi_0 \tag{5.11}$$

in $\operatorname{Ext}^4(A_2)$. We prepare the following lemma which is proved later.

**Lemma 5.12.** $v_2^3 h_{10} h_{11} b_{10}\zeta_2$ *represents a non-trivial element of* $\operatorname{Ext}^5(A_2)$.

Apply $h_{10} \in \operatorname{Ext}^1(A)$ to (5.11) to see $v_2^3 h_{10} b_{10}^2 = -h_{10} b_{11}^2 + kv_2^3 h_{10} h_{11} b_{10}\zeta_2$. We also have $v_2^2 h_{10} b_{10} = h_{11} b_{11}$ and $v_2 h_{11} b_{10} = -h_{10} b_{11}$ in $\operatorname{Ext}^3(A_2)$ by the same fashion as in $\operatorname{Ext}^3(A_1)$, because here there is no indeterminacy, either. Thus we have $v_2^3h_{10}b_{10}^2 = -h_{10}b_{11}^2$ and so $kv_2^3h_{10}h_{11}b_{10}\zeta_2 = 0$. Now Lemma 5.12 implies $k = 0$ as desired. q.e.d.

*Proof of Lemma 5.12.* In the cobar complex, $v_2^3h_{10}h_{11}b_{10}\zeta_2$ is homologous to $-v_1v_2^3b_{10}^2\zeta_2$ by $d(v_2^3t_2b_{10}\zeta_2)$. Therefore, it is a $v_1$-image of $-v_2^3b_{10}^2\zeta_2 \in \mathrm{Ext}^{5,72}(A_1)$. Furthermore, Theorem 5.8 implies $\mathrm{Ext}^{4,76}(A_1) = \{v_2^2b_{11}\xi\}$ and $\delta(v_2^2b_{11}\xi) = 0$ in (5.10), since $h_{11}\xi = 0$ in $\mathrm{Ext}^3(A_1)$. Since $-v_2^3b_{10}^2\zeta_2 \neq 0 \in \mathrm{Ext}^{5,72}(A_1)$, we have the result. q.e.d.

## 6. The homotopy groups of $L_2V_1$

Let $W_k$ denote the cofiber of $\beta_1^k : \Sigma^{10k}S^0 \to S^0$ and put $V_k = W_k \wedge V$ for $V = V(1)$. First we show the following:

**Proposition 6.1.** *For $k < 4$, $W_k$ admits a map $m_k : W_k \wedge W_k \to W_k$ such that $m_k(i \wedge 1_{W_k}) = 1_{W_k}$ for the inclusion $i : S^0 \to W_k$ to the bottom cell.*

*Proof.* It suffices to show $[W_k, W_k]_{10k} = 0$. In fact, we have the cofiber sequence $\Sigma^{10k}W_k \xrightarrow{\beta_1^k} W_k \xrightarrow{i\wedge 1} W_k \wedge W_k$.

The cofiber sequence defining $W_k$ induces the exact sequences

$$\pi_{10k+1}(W_k) \xrightarrow{\beta_1^{k*}} \pi_{20k+1}(W_k) \longrightarrow [W_k, W_k]_{10k} \longrightarrow \pi_{10k}(W_k) \xrightarrow{\beta_1^{k*}}, \quad \text{and}$$

$$\pi_{l-10k}(S^0) \xrightarrow{\beta_{1*}^k} \pi_l(S^0) \longrightarrow \pi_l(W_k) \longrightarrow \pi_{l-10k-1}(S^0) \xrightarrow{\beta_{1*}^k} .$$

From the table [13, Th. B] (*cf.* [20, Table A3.4]) of homotopy groups of spheres, we pick out

$$\pi_{10k}(S^0) = \mathbf{Z}/3\{\beta_1^k\}\ (k < 6), \quad \pi_{10k}(S^0) = 0\ (k = 6, 7, 8),$$
$$\pi_{10k+1}(S^0) = 0\ (k : \text{even} < 8) \quad \text{and} \quad \pi_{-1}(S^0) = 0.$$

Therefore, $\pi_{10k}(W_k) = 0$ for $k \le 8$ and $\pi_{20k+1}(W_k) = 0$ for $k < 4$. q.e.d.

**Lemma 6.2.** $[V_1, V_1]_4 = 0$.

*Proof.* Recall [27, Th. 6.11] the track groups of $V$:

$$[V, V]_{-7} = 0, \quad [V, V]_{-6} = \mathbf{Z}/3\{\delta_0\}, \quad [V, V]_4 = \mathbf{Z}/3\{\beta_1\delta_0\},$$
$$[V, V]_5 = \mathbf{Z}/3\{\beta_1\delta_1\}, \quad [V, V]_{14} = \mathbf{Z}/3\{\beta_1^2\delta_0\} \quad \text{and} \quad [V, V]_{15} = \mathbf{Z}/3\{\beta_1^2\delta_1\}.$$

Observing the exact sequences associating to the cofiber sequence $\Sigma^{10}V \xrightarrow{\beta_1} V \to V_1$, we see that $[V, V_1]_4 = 0 = [V, V_1]_{15}$ and then the lemma follows. q.e.d.

**Proposition 6.3.** *There exists a pairing $\nu_1 : V \wedge V_1 \to V_1$ which is an extension of the identity $1 : V_1 \to V_1$.*

*Proof.* Let $M$ denote the mod 3 Moore spectrum as before, and we have the splitting

$$M \wedge V_1 = V_1 \vee \Sigma V_1,$$

since $M \wedge V = V \vee \Sigma V$. This yields two maps

$$m : M \wedge V_1 \longrightarrow V_1 \quad \text{and} \quad d : \Sigma V_1 \longrightarrow M \wedge V_1$$

as a projection and an inclusion. Consider the exact sequence

$$[V \wedge V_1, V_1]_0 \xrightarrow{i_1^*} [M \wedge V_1, V_1]_0 \xrightarrow{\alpha^*} [M \wedge V_1, V_1]_4.$$

Here $[M \wedge V_1, V_1]_4 = [V_1, V_1]_5$ under the induced map from $d$ by Lemma 6.2. Then we see that $\alpha^*(m) = 0$, since it is the composition $\Sigma^5 V \xrightarrow{d} \Sigma^4 M \wedge V \xrightarrow{\alpha\wedge 1} M \wedge V \xrightarrow{m} V$, which is shown to be trivial in [23, Lemma 2.6]. Thus we have the desired map $\nu_1$ by pulling $m$ back under $i_1^*$. q.e.d.

**Corollary 6.4.** *$V_1$ is a ring spectrum.*

*Proof.* By Proposition 6.1, we have a splitting

$$V_1 \wedge V_1 = (V \wedge V_1) \vee \Sigma^{11}(V \wedge V_1),$$

which defines a map $j : V_1 \wedge V_1 \to V \wedge V_1$. Now the structure map is set to be a composition $\nu_1 j$. q.e.d.

Here by a ring spectrum, we mean a spectrum $X$ together with maps $\mu : X \wedge X \to X$ and $\eta : S^0 \to X$ such that the composition

$$X = S^0 \wedge X \xrightarrow{\eta\wedge 1_X} X \wedge X \xrightarrow{\mu} X$$

is the identity.

We now consider the $E_2$-terms $E_2^*(L_2V_k)$ of the Adams-Novikov spectral sequence for computing $\pi_*(L_2V_k)$.

**Proposition 6.5.** *We have a $K(2)_*$-module isomorphism*

$$E_2^*(L_2V_k) \cong P \otimes F \otimes \Lambda(\zeta_2, g_k)$$

*for $k > 0$. Here $|g_k| = 10k + 1$, and $P$ and $F$ are those of Theorem 5.8.*

*Proof.* By degree reason, one of $E_2^{s,t}(L_2V)$ and $E_2^{s,t-10k-1}(L_2V)$ is trivial. The cofiber sequence $\Sigma^{10k}V \xrightarrow{\beta_1^k} V \to V_k \to \Sigma^{10k+1}V$ gives rise to the split exact sequence of $E_2$-terms

$$0 \longrightarrow E_2^{s,t}(L_2V) \longrightarrow E_2^{s,t}(L_2V_k) \longrightarrow E_2^{s,t-10k-1}(L_2V) \longrightarrow 0. \tag{6.6}$$

Now use Theorem 5.8 to get the proposition. q.e.d.

**Lemma 6.7.** *For an element $x \in P \otimes F \otimes \Lambda(\zeta_2)$,*

$$d_2(xg_1) = b_{10}x \in E_2^*(L_2V_1) \quad \text{and} \quad d_4(xg_2) = b_{10}^2 x \in E_2^*(L_2V_2).$$

*Proof.* Let $xg_k \in E_2^s(L_2V_k)$ for $k = 1, 2$, and consider the cofiber sequence

$$\Sigma^{10k}V \wedge \overline{E}_{s+5} \xrightarrow{\beta_1^k} V \wedge \overline{E}_{s+5} \xrightarrow{i} V_k \wedge \overline{E}_{s+5} \xrightarrow{j} \Sigma^{10k+1}V \wedge \overline{E}_{s+5}.$$

Then it satisfies $E(2)_*(\beta_1^k) = 0$. Furthermore, we see that $j_*(xg_k) = x$ is a permanent cycle of the Adams-Novikov spectral sequence for $\pi_*(V \wedge \overline{E}_{s+3})$, since $d_r(x) = 0$ for $r < 5$ by degree reason and for $r > 4$ by Lemma 1.2. By Lemma 1.4,

$$d_{2k}(xg_k) = i_*(\overline{\beta_1^k x}) = i_*(b_{10}^k x) = b_{10}^k x \quad \text{in } E_{2k}^{s+2k}(V_k \wedge \overline{E}_{s+5}),$$

since $\beta_1$ represents $b_{10}$. Since $L_2V_k = \lim_s V_k \wedge \overline{E}_s$, we have a map $L_2V_k \to V_k \wedge \overline{E}_{s+5}$ and so we see the lemma by the naturality of the differential. q.e.d.

**Theorem 6.8.** *$\pi_*(L_2V_1) \cong F \otimes \Lambda(\zeta_2)$ as a $K(2)_*$-module.*

*Proof.* The short exact sequence (6.6) gives rise to the long exact sequence

$$\cdots \longrightarrow E_3^s(L_2V) \xrightarrow{\delta} E_3^{s+2}(L_2V) \longrightarrow E_3^{s+2}(L_2V_1) \longrightarrow E_3^{s+2}(L_2V) \xrightarrow{\delta} \cdots,$$

since the $E_3^*$-term is a homology of the complex $(E_2^*, d_2)$. By the definition of the connecting homomorphism $\delta$, Lemma 6.7 implies

$$\delta(x) = b_{10}x.$$

Therefore, we obtain

$$E_3^*(L_2V_1) = F \otimes \Lambda(\zeta_2),$$

which is described in a chart as follows:

| $s$ \ $t-s$ mod 16 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | | | | | | | | $b_{11}\xi\zeta_2$ | | | | | | | | |
| 4 | | | | | $\psi_1\zeta_2$ | | | | $b_{11}\xi$ | | | | $\psi_0\zeta_2$ | | | |
| 3 | | $b_{11}\zeta_2$ | | | | $\xi\zeta_2$ $\psi_1$ | | | | | | | | $\psi_0$ | | |
| 2 | | | $h_{10}\zeta_2$ $b_{11}$ | | | | $\xi$ | | | | $h_{11}\zeta_2$ | | | | | |
| 1 | | | | $h_{10}$ | | | | | | | | $h_{11}$ | | | | $\zeta_2$ |
| 0 | 1 | | | | | | | | | | | | | | | |

This chart shows that $E_3^*(L_2V_1) = E_\infty^*(L_2V_1)$ and no extension problem arises. Therefore, $\pi_*(L_2V_1) = E_3^*(L_2V_1)$ as desired. q.e.d.

Using this theorem, we obtain one of the main lemmas for determining the homotopy groups of $L_2V(1)$:

**Lemma 6.9.** *In the Adams-Novikov spectral sequence $\{E_r^*(L_2V)\}$ for computing $\pi_*(L_2V)$, let $x \in E_r^s(L_2V)$ denote an element belonging to the subquotient of $P \otimes F$ $(V = V(1))$. Then,*

$$d_r(x\zeta_2) = y\zeta_2 \in E_r^s(L_2V)$$

*for some $y \in E_r^s(L_2V)$ corresponding to $y \in P \otimes F \subset E_2^*(L_2V)$.*

*Proof.* Theorem 6.8 certifies the existence of the homotopy element $\zeta \in \pi_*(L_2V_1)$ detected by $\zeta_2$. Consider the composition $z : V = V \wedge S^0 \xrightarrow{1\wedge\zeta} \Sigma V \wedge V_1 \xrightarrow{\nu_1} \Sigma V_1$ and the cofiber sequence

$$\Sigma^{-1}V \xrightarrow{z} V_1 \xrightarrow{i} C_z \xrightarrow{j} V.$$

Then $E(2)_*(z) = 0$, since $E(2)_*(V) = K(2)_*$ and $E(2)_*(V_1) = K(2)_* \otimes \Lambda(g_1)$, and so we obtain the short exact sequence

$$0 \longrightarrow K(2)_* \otimes \Lambda(g_1) \xrightarrow{i_*} E(2)_*(C_z) \xrightarrow{j_*} K(2)_* \longrightarrow 0.$$

This associates the long exact sequence of $E_2$-terms:

$$\cdots \longrightarrow P \otimes F \otimes \Lambda(\zeta_2) \xrightarrow{\zeta_2} P \otimes F \otimes \Lambda(\zeta_2, g_1) \xrightarrow{i_*} E_2^*(L_2C_z) \xrightarrow{j_*} P \otimes F \otimes \Lambda(\zeta_2) \longrightarrow \cdots,$$

which gives us

$$E_2^*(L_2C_z) = i_*(P \otimes F \oplus P \otimes F \otimes \Lambda(\zeta_2)g_1) \oplus j_*^{-1}(P \otimes F\zeta_2).$$

By Lemma 6.7, we compute

$$d_2(i_*(xg_1)) = i_*d_2(xg_1) = i_*(b_{10}x) \in i_*(P \otimes F) \subset E_2^*(L_2C_z)$$

for $x \in P \otimes F$. Therefore,

$$E_3^*(L_2C_z) = i_*(F \oplus P \otimes F\zeta_2g_1) \oplus j_*^{-1}(P \otimes F\zeta_2).$$

For $x\zeta_2 \in E_r^s(L_2V)$ survived from $P \otimes F\zeta_2 \subset E_2^s(L_2V)$, we compute

$$\begin{aligned} d_r(x\zeta_2) &= d_r(j_*(x\zeta_2)) = j_*(d_r(x\zeta_2)) \\ &\in (\text{the subquotient of } j_*(E_3^{s+r}(L_2C_z))) \\ &= (\text{the subquotient of } P \otimes F\zeta_2) \subset E_r^{s+r}(L_2V). \quad \text{q.e.d.} \end{aligned}$$

From here on in this section, we will consider the duality. Let $u \in \pi_*(L_2V_1)$ denote the homotopy element detected by the element $b_{11}\xi\zeta_2 \in E_\infty(L_2V_1)$. Then we have an element $u^* \in [V_1, I_2]_{-39} = \mathrm{Hom}(\pi_{39}(L_2V_1), \mathbf{Q}/\mathbf{Z}_{(3)})$ dual to $u$. Here $I_2$ denotes the Brown-Comenetz dual of $L_2S^0$. Consider also a composition

$$\overline{u} : \Sigma^{-39}V_1 \wedge V_1 \xrightarrow{\mu} \Sigma^{-39}V_1 \xrightarrow{u^*} I_2.$$

Consider the adjoint

$$u^{\#} : \Sigma^{-39}V_1 \longrightarrow \mathrm{Map}(V_1, I_2)$$

of this map $\overline{u}$. Here $\mathrm{Map}(G, I_2) = I_2G$ is the Brown-Comenetz dual of $G$, which indicates $\pi_*(I_2G) = \mathrm{Hom}(\pi_*(G), \mathbf{Q}/\mathbf{Z}_{(3)})$. The computation of the homotopy groups, Theorem 6.8 and Proposition 5.9, show the following

**Corollary 6.10.** *The adjoint map $L_2u^{\#}$ is homotopy equivalent.*

In fact, the chart in Theorem 6.8 is written as follows using the dual elements $x^*$ such that $xx^* = b_{11}\xi\zeta_2$:

| $s$ ↑ | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | | | | | | | | $1^*$ | | | | | | | | |
| 4 | | | | | $h_{10}^*$ | | | | $\zeta_2^*$ | | | | $h_{11}^*$ | | | |
| 3 | | $\xi^*$ | | | | $b_{11}^*$ $h_{10}\zeta_2^*$ | | | | | | | | $h_{11}\zeta_2^*$ | | |
| 2 | | | $h_{10}\zeta_2$ $b_{11}$ | | | | $\xi$ | | | | $h_{11}\zeta_2$ | | | | | |
| 1 | | | | $h_{10}$ | | | | | | | | $h_{11}$ | | | | $\zeta_2$ |
| 0 | $1$ | | | | | | | | | | | | | | | |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |

$t - s$ mod 16 ⟶

## 7. $\mathbf{Z}/3[v_2^9, v_2^{-9}]$-MODULE STRUCTURE

In this section, we will show the following proposition which will give the homotopy groups $\pi_*(L_2V(1))$ a $\mathbf{Z}/3[v_2^9, v_2^{-9}]$-module structure.

**Lemma 7.1.** *In the Adams-Novikov spectral sequence for computing $\pi_*(L_2V(1))$, if $d_r(x)b_{10} = 0$, then $d_r(x) = 0$.*

*Proof.* Consider the cofiber sequence

$$\Sigma^{10}V \xrightarrow{\beta_1} V \xrightarrow{i_1} V_1 \longrightarrow \Sigma^{11}V$$

for $V = V(1)$. As in (6.6), we have the short exact sequence

$$0 \longrightarrow E_2^{s,t}(L_2V) \xrightarrow{i_{1*}} E_2^{s,t}(L_2V_1) \longrightarrow E_2^{s,t-11}(L_2V) \longrightarrow 0.$$

Since $d_2(xg_1) = b_{10}x$ in $E_2^{s,t}(L_2V_1)$ by Lemma 6.7, the short exact sequence yields the long one:

$$\begin{aligned} \cdots \longrightarrow E_3^{s-2,t-12}(L_2V) &\xrightarrow{b_{10}} E_3^{s,t}(L_2V) \\ &\xrightarrow{i_{1*}} E_3^{s,t}(L_2V_1) \longrightarrow E_3^{s,t-11}(L_2V) \longrightarrow \cdots . \end{aligned}$$

Suppose that $d_r(x) \neq 0$. Then by Theorem 5.8, the equation $d_r(x)b_{10} = 0$ in $E_r^*(L_2V)$ implies that $d_r(x)b_{10}$ must be killed. Let $y$ denote the killer, and $i_{1*}(y)$ is a permanent cycle by Theorem 6.8. Notice that any element in $E_2^s(L_2V)$ for $s > 5$

is divided by $b_{10}$. By a diagram chasing with Lemma 6.7, we see that $xb_{10}$ is a permanent cycle. Therefore, we have $i_{1*}(xb_{10}) = i_{1*}(y)$ in the homotopy groups $\pi_*(L_2V_1)$. By Theorem 6.8, the Adams-Novikov filtration of $i_{1*}(y)$ is less than 6 if $i_*(y)$ survives to $E_r^*(L_2V_1)$. Since $y$ has a greater filtration than that of $xb_{10}$, the difference is no less than 4 by degree reason. Therefore, $\mathrm{filt}^{AN}\ xb_{10} \le \mathrm{filt}^{AN}\ i_{1*}(y) - 4 < 6 - 4 = 2$, which contradicts the fact that $\mathrm{filt}^{AN}\ b_{10} = 2$. So suppose that $i_*(y)$ is killed. Then, $i_*(y) = zb_{10} = d_2(zg_1)$ for some $z$ by Lemma 6.7 and Theorem 6.8. Then by a diagram chasing in the Adams-Novikov exact couple, we see that $z = x$, which also contradicts our hypothesis. Therefore $d_r(x) = 0$ as desired. q.e.d.

The following chart may help to understand the above proof.

$$\begin{array}{c|c|c|c|c}
d_r(x) & \to & d_r(x)b_{10} & & \\
 & & \nwarrow d_s & & \\
\nwarrow d_r & & y & \to & i_{1*}(y) \\
 & & & \nearrow & \\
x & \to & xb_{10} & & \\
\hline
V & \xrightarrow{\beta_1} & V & \xrightarrow{i_1} & V_1
\end{array}$$

**Proposition 7.2.** *In the Adams-Novikov spectral sequence for computing* $\pi_*(L_2V(1))$,

$$d_r(v_2^{9t}x) = v_2^{9t}d_r(x)$$

*for integers* $t \ge 0$ *and* $r > 1$ *and for a class* $x$ *of* $E_r^*(V(1))$.

*Proof.* We proceed by induction on $t$. It is trivial for $t = 0$. Suppose that $d_r(v_2^{9t}x) = v_2^{9t}d_r(x)$. By Lemma 2.4, $v_2^3b_{11}$ is a permanent cycle, and so is $(v_2^3b_{11})^2 = v_2^6b_{11}^2 = v_2^9b_{10}^2$. Therefore, we have

$$v_2^{9t}d_r(x)(v_2^9b_{10}^2) = d_r(v_2^{9t}x)(v_2^9b_{10}^2) = d_r(v_2^{9t+9}xb_{10}^2) = d_r(v_2^{9t+9}x)b_{10}^b$$

by (1.7). Now apply Lemma 7.1 to see $d_r(v_2^{9t+9}x) = v_2^{9t+9}d_r(x)$. q.e.d.

## 8. The Adams-Novikov differential on $L_2V(1)$

In this section, we compute the differential $d_5$ of the Adams-Novikov spectral sequence for computing $\pi_*(L_2V)$, using the so-called Toda differential. Recall [25] the Toda differential:

$$d_5(b_{11}) = \lambda h_{10}b_{10}^3 \quad \text{in } E_5^7(S^0) = E_2^7(S^0) \tag{8.1}$$

for some non-zero element $\lambda \in \mathbf{Z}/3$. Consider the composition $f : S^0 \to V(1) \to L_2V(1)$ of the inclusion to the bottom cell and the localization map.

**Lemma 8.2.** *In* $E_5^9(L_2V)$,

$$f_*d_5(b_{11}^2) = \lambda v_2h_{11}b_{10}^4$$

*for the induced map* $f_* : E_5^*(S^0) \to E_5^*(L_2V)$.

*Proof.* By the derivation property and (8.1),

$$d_5(b_{11}^2) = -\lambda h_{10}b_{10}^3b_{11},$$

which is sent to $\lambda v_2h_{11}b_{10}^4$ by the map $f_*$, since $h_{10}b_{11} = -v_2h_{11}b_{10}$ by Proposition 5.9. q.e.d.

**Lemma 8.3.** *If $d_5(v_2^j) = xv_2^{j-2}h_{11}b_{10}^2$ for some $x \in \mathbf{Z}/3$ in $E_5^5(L_2V)$, then*

$$\begin{array}{rcll} d_5(v_2^{j+3}) & = & (x-\lambda)v_2^{j+1}h_{11}b_{10}^2 & \text{in } E_5^5(L_2V), \\ d_5(v_2^j h_{10}) & = & 0 & \text{in } E_5^6(L_2V), \\ d_5(v_2^j h_{11}) & = & 0 & \text{in } E_5^6(L_2V) \quad \text{and} \\ d_5(v_2^{j+3}b_{11}) & = & xv_2^{j+3}h_{10}b_{10} & \text{in } E_5^7(L_2V). \end{array}$$

*Proof.* We compute first:

$$\begin{array}{rcll} d_5(v_2^{j+3}b_{10}^2) & = & -d_5(v_2^j b_{11}^2) & \text{by } b_{11}^2 = -v_2^3 b_{10}^2 \text{ in Proposition 5.9}, \\ & = & -d_5(v_2^j)b_{11}^2 - v_2^j f_* d_5(b_{11}^2) & \text{by (1.7)}, \\ & = & -xv_2^{j-2}h_{11}b_{10}^2 - \lambda v_2^{j+1}h_{11}b_{10}^4 & \text{by the hypothesis and Lemma 8.2}. \end{array}$$

Since $b_{10}$ acts monomorphically by Theorem 5.8, we obtain the first equation of the lemma. The other equations follow from Lemma 2.4, using the hypothesized equation, the derivation property and Proposition 5.9. q.e.d.

Now Theorem 2.6, Proposition 7.2 and Lemma 8.3 yield

**Proposition 8.4.** *The differential $d_5 : E_5^*(L_2V) \to E_5^{*+5}(L_2V)$ for $v_2^j$, $v_2^j h_{10}$, $v_2^j h_{11}$ and $v_2^j b_{11}$ is given as follows:*

$$d_5(v_2^j) = \begin{cases} 0, & j \equiv 0,1,5 \ (9), \\ -\lambda v_2^{j-2}h_{11}b_{10}^2, & j \equiv 3,4,8 \ (9), \\ \lambda v_2^{j-2}h_{11}b_{10}^2, & j \equiv 2,6,7 \ (9), \end{cases}$$

$$d_5(v_2^j h_{10}) = 0,$$

$$d_5(v_2^j h_{11}) = 0,$$

$$d_5(v_2^j b_{11}) = \begin{cases} \lambda v_2^j h_{10}b_{10}^3, & j \equiv 0,1,5 \ (9), \\ 0, & j \equiv 3,4,8 \ (9), \\ -\lambda v_2^j h_{10}b_{10}^3, & j \equiv 2,6,7 \ (9). \end{cases}$$

Lemma 1.4 together with this proposition immediately implies the following corollary, since $v_2$ detects a homotopy element of $\pi_*(L_2V)$.

**Corollary 8.5.** *Let $x \in E_5^*(L_2V)$ be a permanent cycle with $\delta(v_2x) \neq 0$. Then*

$$d_5(v_2x) = \lambda\delta(x)b_{10}^2 + \mu\delta(x)h_{11}\zeta_2b_{10} \quad (\mu \in \mathbf{Z}/3),$$

*where $\delta$ denotes the Bockstein operator given in* (1.8).

*Proof.* By the definition of $\delta$, $\delta(v_2) = h_{11}$ is seen by Landweber's formula $\eta_R(v_2) \equiv v_2 + v_1t_1^3 \bmod (3, v_1^2)$. Since $v_2$ is a permanent cycle of the Adams-Novikov spectral sequence, Lemma 1.9 shows $d_5(v_2x) = i_{1*}(\omega(v_2))\delta(x)$. We also have $d_5(v_2^2) = \lambda h_{11}b_{10}^2$ by Proposition 8.4. Comparing these, we obtain $i_{1*}(\omega(v_2)) \equiv \lambda b_{10}^2 \bmod \operatorname{Ker} h_{11}$. Proposition 5.9 and Theorem 5.8 show that $\operatorname{Ker} h_{11} = \mathbf{Z}/3\{h_{11}\zeta_2b_{10}\}$ at the degree $|b_{10}^2| = 20$. q.e.d.

**Lemma 8.6.** *Suppose that $d_5(v_2^j\psi_0) = xv_2^{j-3}\xi b_{11}b_{10}^2$ for some $x \in \mathbf{Z}/3$ in $E_5^8(L_2V)$. Then,*

$$\begin{array}{rcll} d_5(v_2^{j+3}\psi_0) &=& (x-\lambda)v_2^j\xi b_{11}b_{10}^2 & \text{in } E_5^8(L_2V), \\ d_5(v_2^j\xi) &=& 0 & \text{in } E_5^7(L_2V), \\ d_5(v_2^{j-1}b_{11}\xi) &=& 0 & \text{in } E_5^9(L_2V) \quad \text{and} \\ d_5(v_2^{j+4}\psi_1) &=& xv_2^{j+3}\xi b_{10}^3 & \text{in } E_5^8(L_2V). \end{array}$$

*Proof.* It follows from a direct computation using (1.7), Lemma 8.2 and Proposition 5.9 that

$$\begin{aligned} d_5(v_2^{j+3}\psi_0 b_{10}^2) &= -d_5(v_2^j\psi_0 b_{11}^2) \\ &= -d_5(v_2^j\psi_0)b_{11}^2 - v_2^j\psi_0 d_5(b_{11}^2) \\ &= -xv_2^{j-3}\xi b_{11}b_{10}^2 b_{11}^2 - \lambda v_2^{j+1}\psi_0 h_{11}b_{10}^4 \\ &= xv_2^j\xi b_{11}b_{10}^4 + \lambda v_2^j\xi b_{11}b_{10}^4 \\ &= (x-\lambda)v_2^j\xi b_{11}b_{10}^4. \end{aligned}$$

This implies the first equation.

The second and third ones follow from the hypothesized equation, since $h_{10}\psi_0 = -b_{10}\xi$, $h_{11}\psi_0 = v_2^{-1}b_{11}\xi$ and $h_{1k}\xi = 0$ for $k = 0, 1$ by Proposition 5.9.

The last one is also obtained from the similar computation:

$$\begin{aligned} d_5(v_2^{j+4}b_{10}\psi_1) &= -d_5(v_2^{j+3}b_{11}\psi_0) \\ &= -d_5(v_2^j\psi_0)v_2^3 b_{11} \quad \text{by Lemma 2.4} \\ &= xv_2^{j+3}\xi b_{10}^4, \end{aligned}$$

using the relations in Proposition 5.9. q.e.d.

In the same way as above, we obtain the following:

**Lemma 8.7.** *Suppose that $d_5(v_2^j\zeta_2) = xv_2^{j-2}h_{11}b_{10}^2\zeta_2$ in $E_5^6(L_2V)$ and $d_5(v_2^j\psi_0) = x'v_2^{j-3}\xi b_{11}b_{10}^2\zeta_2$ in $E_5^9(L_2V)$ for some $x, x' \in \mathbf{Z}/3$. Then,*

a)
$$\begin{array}{rcll} d_5(v_2^{j+3}\zeta_2) &=& (x-\lambda)v_2^{j+1}h_{11}b_{10}^2\zeta_2 & \text{in } E_5^6(L_2V), \\ d_5(v_2^j h_{10}\zeta_2) &=& 0 & \text{in } E_5^7(L_2V), \\ d_5(v_2^j h_{11}\zeta_2) &=& 0 & \text{in } E_5^7(L_2V) \quad \text{and} \\ d_5(v_2^{j+3}b_{11}\zeta_2) &=& xv_2^{j+3}h_{10}b_{10}^3\zeta_2 & \text{in } E_5^8(L_2V). \end{array}$$

b)
$$\begin{array}{rcll} d_5(v_2^{j+3}\psi_0\zeta_2) &=& (x'-\lambda)v_2^j\xi b_{11}b_{10}^2\zeta_2 & \text{in } E_5^7(L_2V), \\ d_5(v_2^j\xi\zeta_2) &=& 0 & \text{in } E_5^8(L_2V), \\ d_5(v_2^{j-1}b_{11}\xi\zeta_2) &=& 0 & \text{in } E_5^{10}(L_2V) \quad \text{and} \\ d_5(v_2^{j+4}\psi_1\zeta_2) &=& x'v_2^{j+3}\xi b_{10}^3\zeta_2 & \text{in } E_5^9(L_2V). \end{array}$$

**Lemma 8.8.** *The hypothesis of the above lemma is satisfied. That is, $d_5(v_2^j\zeta_2) = xv_2^{j-2}h_{11}b_{10}^2\zeta_2$ and $d_5(v_2^j\psi_0\zeta_2) = x'v_2^{j-3}\xi b_{11}b_{10}^2\zeta_2$ for $j \in \mathbf{Z}$ and $x, x' \in \mathbf{Z}/3$.*

*Proof.* By Theorem 5.8, we may put

$$d_5(v_2^j\zeta_2) = xv_2^{j-2}h_{11}b_{10}^2\zeta_2 + yv_2^{j-2}\psi_1\zeta_2 b_{10} + zv_2^{j-2}b_{10}^3$$

for some $x, y, z \in \mathbf{Z}/3$ by comparing degrees. By Lemma 2.4, $h_{11}$ sends this to

$$d_5(v_2^j h_{11}\zeta_2) = yv_2^{j-1}\xi\zeta_2 b_{10}^2 + zv_2^{j-2}h_{11}b_{10}^3,$$

since $h_{11}\psi_1 = v_2\xi b_{10}$ by Proposition 5.9. On the other hand, Proposition 8.4 says $d_5(v_2^{j-1}) = wv_2^{j-3}h_{11}b_{10}^2$ for some $w \in \mathbf{Z}/3$. The element $\beta_2 = v_2h_{11}\zeta_2$ sends this to

$$d_5(v_2^jh_{11}\zeta_2) = 0$$

by (1.7). Therefore we have $y = 0 = z$ as desired.

For the other equations, it follows immediately from Theorem 5.8 and Lemma 6.9. q.e.d.

## 9. The homotopy groups of $L_2V_2$ and $L_2V_3$

In this section, we will determine the homotopy groups of $L_2V_2$ and $L_2V_3$ for $V_k = V \cup_{\beta_1^k} C\Sigma^{10k}V$ ($V = V(1)$). Recall from Theorem 5.8 the $K(2)_*$-module

$$F = K(2)_*\{1, h_{10}, h_{11}, b_{11}, \xi, \psi_0, \psi_1, b_{11}\xi\}.$$

Our first result in this section is:

**Theorem 9.1.** *$\pi_*(L_2V_2) \cong F \otimes \Lambda(\zeta_2, b_{10})$ as a $K(2)_*$-module.*

*Proof.* Consider the Adams-Novikov spectral sequence $\{E_r^*(L_2V_2)\}$. By (6.6) and Lemma 6.7, we have the long exact sequence

$$\begin{aligned}(9.2)\qquad \cdots \longrightarrow E_5^s(L_2V) &\xrightarrow{\delta} E_5^{s+4}(L_2V)\\ &\xrightarrow{i_*} E_5^{s+4}(L_2V_2) \longrightarrow E_5^{s+4}(L_2V) \xrightarrow{\delta} \cdots\end{aligned}$$

with $\delta(x) = b_{10}^2x$. Therefore,

$$E_5^*(L_2V_2) = F \otimes \Lambda(\zeta_2, b_{10}),$$

and we have the chart:

| $s$ ↑ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 7 | | $1^*$ | | | | | | | |
| 6 | | | $\zeta_2^*$ | | | | $h_{11}^*$ | | |
| 5 | | | | | | | | $b_{10}^*$ $h_{11}\zeta_2^*$ | |
| 4 | $b_{11}\zeta_2^*$ | | | | $b_{10}h_{10}^*$ $\psi_0^*$ | | | | $b_{10}\zeta_2^*$ |
| 3 | | $b_{11}\zeta_2$ | | | | $\xi\zeta_2$ $b_{10}h_{11}$ $\psi_1$ | | | |
| 2 | | | $h_{10}\zeta_2$ $b_{11}$ | | | | $\xi$ | | |
| 1 | | | | $h_{10}$ | | | | | |
| 0 | $1$ | | | | | | | | |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

$t - s$ mod 16 ⟶

| $s$ ↑ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 7 | | | | | | | | |
| 6 | | | | | | | $h_{10}^*$ | |
| 5 | | | | $\xi^*$ | | | | $b_{11}^*$ $h_{10}\zeta_2^*$ |
| 4 | $b_{10}\zeta_2^*$ | | | | $b_{10}h_{11}^*$ $\psi_1^*$ $\xi\zeta_2^*$ | | | |
| 3 | | $b_{10}\zeta_2$ | | | | $\psi_0$ $b_{10}h_{10}$ | | |
| 2 | | | $h_{11}\zeta_2$ $b_{10}$ | | | | | |
| 1 | | | | $h_{11}$ | | | | $\zeta_2$ |
| 0 | | | | | | | | |
| | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |

$t - s$ mod 16 ⟶

Here $x^*$ denotes an element such that $xx^* = b_{10}b_{11}\xi\zeta_2$. The element $x$ can be read off from Proposition 5.9. By this chart, $d_5(x) = 0$ unless $x \in F' = K(2)_*\{1, h_{10}, \zeta_2, h_{10}\zeta_2, b_{11}\}$. Proposition 8.4, Lemmas 8.7 and 8.8 show $d_5(x) = 0$ for $x \in F'$ in $E_5^*(L_2V_2)$, since $i_*(xb_{10}^2) = 0$ in (9.2). Therefore we see that

$E_5^*(L_2V_2) = E_\infty(L_2V_2)$. Besides, the extension is trivial, since $L_2V_2$ is an $M$-module spectrum. q.e.d.

Now turn to the homotopy groups $\pi_*(L_2V_3)$. By Theorem 5.8 and Proposition 6.5, we see that $E_2^*(L_2V_3) = E_2^*(L_2V) \oplus E_2^*(L_2V)g_3$. Therefore, by degree reason, we see the following

**Lemma 9.3.** $E_6^*(L_2V_3) = E_6^*(L_2V) \oplus E_6^*(L_2V)g_3$.

In the same way as Lemma 6.7, we obtain the following

**Lemma 9.4.** *For $x \in P \otimes F \otimes \Lambda(\zeta_2)$ with $d_5(x) = 0$,*

$$d_6(xg_3) = b_{10}^3 x \in E_6^*(L_2V_3).$$

**Lemma 9.5.** *Let $wg_3 \in E_2^*(L_2V_3)$ denote a non-zero element with $w \in P \otimes F \otimes \Lambda(\zeta_2)$. If $d_r(x) = wg_3 \neq 0$, then $r = 5$.*

*Proof.* Note that $d_s(wg_3) = 0$ for $s \geq 2$ by the assumption that $d_r(x) = wg_3$. If $r > 5$, $wg_3$ survives out of $d_5$, and so Lemma 9.4 shows that $d_6(wg_3) = b_{10}^3 w$ in $E_6^*(L_2V_3)$. Therefore $b_{10}^3 w$ must be 0 in $E_6^*(L_2V_3)$, that is, there exists an element $y \in E_5^*(L_2V_3) = E_2^*(L_2V_3)$ such that $d_5(y) = b_{10}^3 w$ in $E_5^*(L_2V_3)$. If $y = y_0 b_{10}^u$ with $y_0 \in F \otimes \Lambda(\zeta_2)$ and $u \geq 0$, then $d_5(y_0) = b_{10}^3 w_0$ for $w_0$ with $w = b_{10}^u w_0$, since $b_{10}$ acts monomorphically on $E_5^*(L_2V_3)$ by Theorem 5.8 and the naturality of the Adams-Novikov differential. In fact, $b_{10}$ detects the homotopy element $\beta_1$ by Lemma 2.4. The assumption $wg_3 \neq 0$ in $E_r^*(L_2V_3)$ shows that $u < 3$, since $d_5(y_0 b_{10}^{u-3} g_3) = wg_3$ otherwise, which is verified by sending it under the map $V_3 \to \Sigma^{31}V$ by using the naturality $d_r(\chi b_{10}) = d_r(\chi) b_{10}$. Recall (2.7) the definition of the Adams-Novikov filtration filt and note that filt $y_0 \leq 5$, since $y_0 \in F \otimes \Lambda(\zeta_2)$. Then filt $b_{10}^3 w_0 \leq 10$, and so filt $wg_3 \leq 8$. On the other hand, if $r > 5$, then $r > 8$ by degree reason, and filt $wg_3 > 8$. This is a contradiction. q.e.d.

**Lemma 9.6.** *If $w \in b_{10}^3(P \otimes F \otimes \Lambda(\zeta_2)) \subset E_2^*(L_2V_3)$, then $w$ does not survive to $E_7^*(L_2V_3)$.*

*Proof.* Put $w = b_{10}^3 x$ for $x \in P \otimes F \otimes \Lambda(\zeta_2)$. If $d_5(x) \neq 0$, then $d_5(w) \neq 0$ and $w$ dies. Otherwise, Lemma 9.4 shows $d_6(xg_3) = w$ and $w$ is killed. q.e.d.

**Lemma 9.7.** $E_7^*(L_2V_3) \cong E_\infty^*(L_2V_3)$.

*Proof.* Suppose that $d_r(x) = y$ for $r \geq 7$. If $y \neq 0$, then $y = y_0 b_{10}^u$ with $y_0 \in F \otimes \Lambda(\zeta_2)$ and $u < 3$ by Lemmas 9.5 and 9.6. By degree reason, $r \geq 9$, and filt $y \leq 9$. Therefore, the only possibility supporting a non-trivial differential is $d_9(v_2^t) = v_2^u b_{10}^2 b_{11} \xi \zeta_2$, which does not happen by degree reason. In fact, $|d_9(v_2^t)| \equiv 15$ (9) and $|v_2^u b_{10}^2 b_{11} \xi \zeta_2| \equiv 11$ (9). Therefore, $d_r(x) = 0$ for $r \geq 7$. q.e.d.

*Remark* 9.8. For an element $x \in P \otimes F \otimes \Lambda(\zeta_2) \subset E_2^*(L_2V_3)$, $x$ is a permanent cycle if $d_5(x) = 0$.

**Proposition 9.9.** *In the Adams-Novikov spectral sequence for $L_2V_3$, there exists an element $l \in \mathbf{Z}/3$ such that $d_5(v_2^{3l}\zeta_2) = 0$. Furthermore,*

$$\begin{aligned}
d_5(v_2^{j+3l}\zeta_2) &= \begin{cases} 0, & j \equiv 0,1,5\ (9), \\ -\lambda v_2^{j+3l-2}h_{11}b_{10}^2\zeta_2, & j \equiv 3,4,8\ (9), \\ \lambda v_2^{j+3l-2}h_{11}b_{10}^2\zeta_2, & j \equiv 2,6,7\ (9), \end{cases} \\
d_5(v_2^{j+3l}h_{10}\zeta_2) &= 0, \\
d_5(v_2^{j+3l}h_{11}\zeta_2) &= 0, \\
d_5(v_2^{j+3l}b_{11}\zeta_2) &= \begin{cases} \lambda v_2^{j+3l}h_{10}b_{10}^3\zeta_2, & j \equiv 0,1,5\ (9) \\ 0, & j \equiv 3,4,8\ (9) \\ -\lambda v_2^{j+3l}h_{10}b_{10}^3\zeta_2, & j \equiv 2,6,7\ (9). \end{cases}
\end{aligned}$$

*Proof.* Suppose that $d_5(\zeta_2) = xv_2^{-2}h_{11}b_{10}^2\zeta_2$ in $E_2^*(L_2V)$ by Lemma 8.8. Then the coefficients of $d_5(v_2^3\zeta_2)$ and $d_5(v_2^6\zeta_2)$ are $x - \lambda$ and $x + \lambda$, respectively, by Lemma 8.7. Since $\lambda = \pm 1$, one of $x$, $x - \lambda$ and $x + \lambda$ is zero as desired.

It suffices to show the equation on $d_5(v_2^j\zeta_2)$ by Lemma 8.7. Consider the inclusion $V \to V_3$, and Theorem 2.6 shows that $v_2^j$ is permanent if $j \equiv 0,1,5\ (9)$ in the Adams-Novikov spectral sequence for $L_2V_3$ as well. Take $l \in \mathbf{Z}/3$ so that $d_5(v_2^{3l}\zeta_2) = 0$ as shown above. Then Lemma 8.7 shows the case for $j \equiv 0,3,6\ (9)$. The Bockstein operation $\delta$ acts also trivially on $v_2^{3l}\zeta_2$, in fact, $\delta(v_2^{3l}) = 0$ and $\delta(\zeta_2) = 0$. Now we can apply Lemma 1.9 to get

$$d_5(v_2^{j+3l}\zeta_2) = i_{1*}(\omega(v_2^j))\delta(v_2^{3l}\zeta_2) = 0$$

for $j \equiv 1,5\ (9)$, since $\delta(v_2^{3l}\zeta_2) = 0$ and $\partial(v_2^{j+3l}\zeta_2) \neq 0$ in these cases. The others follow again from Lemma 8.7 immediately. q.e.d.

**Proposition 9.10.** *In the Adams-Novikov spectral sequence for $L_2V_3$, there exists an element $k \in \mathbf{Z}/3$ such that $d_5(v_2^{3k}\psi_0) = 0$. Using $k$ we can state the differential as follows:*

$$\begin{aligned}
d_5(v_2^{3k+j}\psi_0) &= \begin{cases} 0, & j \equiv 0,4,8\ (9), \\ -\lambda v_2^{3k+j-3}\xi b_{11}b_{10}^2, & j \equiv 2,3,7\ (9), \\ \lambda v_2^{3k+j-3}\xi b_{11}b_{10}^2, & j \equiv 1,5,6\ (9), \end{cases} \\
d_5(v_2^j\xi) &= 0, \\
d_5(v_2^j b_{11}\xi) &= 0, \\
d_5(v_2^{3k+j}\psi_1) &= \begin{cases} 0, & j \equiv 3,4,8\ (9), \\ -\lambda v_2^{3k+j-1}\xi b_{10}^3, & j \equiv 2,6,7\ (9), \\ \lambda v_2^{3k+j-1}\xi b_{10}^3, & j \equiv 0,1,5\ (9), \end{cases} \\
d_5(v_2^{3k+3l+j}\psi_0\zeta_2) &= \begin{cases} 0, & j \equiv 0,4,8\ (9), \\ -\lambda v_2^{3k+3l+j-3}\xi b_{11}b_{10}^2\zeta_2, & j \equiv 2,3,7\ (9), \\ \lambda v_2^{3k+3l+j-3}\xi b_{11}b_{10}^2\zeta_2, & j \equiv 1,5,6\ (9), \end{cases} \\
d_5(v_2^j\xi\zeta_2) &= 0, \\
d_5(v_2^j b_{11}\xi\zeta_2) &= 0, \\
d_5(v_2^{3k+3l+j}\psi_1\zeta_2) &= \begin{cases} 0, & j \equiv 3,4,8\ (9), \\ -\lambda v_2^{3k+3l+j-1}\xi b_{10}^3\zeta_2, & j \equiv 2,6,7\ (9), \\ \lambda v_2^{3k+3l+j-1}\xi b_{10}^3\zeta_2, & j \equiv 0,1,5\ (9), \end{cases}
\end{aligned}$$

*for $l \in \mathbf{Z}/3$ of Proposition 9.9.*

*Proof.* Let $\delta$ denote the Bockstein operation in (1.8). By degree reason, $\psi_0^3 = v_2^2\psi_0$ in the cobar complex, and so $0 = \delta(\psi_0^3) = v_2^2\delta(\psi_0) - v_2h_{11}\psi_0 = v_2^2\delta(\psi_0) - b_{11}\xi$. Thus

$$\delta(\psi_0) = v_2^{-2}b_{11}\xi. \tag{9.11}$$

This implies

$$\delta(v_2^2\psi_0) = 0 \tag{9.12}$$

by Proposition 5.9. As in the proof of Proposition 9.9, we obtain an element $k$ of $\mathbf{Z}/3$ such that $d_5(v_2^{3k}\psi_0) = 0$ from Lemma 8.6. Now Corollary 8.5 is applied to show

$$d_5(v_2^{3k+1}\psi_0) = \lambda v_2^{3k-2}b_{11}\xi b_{10}^2$$

under (9.11), since $h_{11}\xi = 0$ by Proposition 5.9. Therefore, we have the proposition for $3k + j$ with $j \equiv 0, 3, 6, 1, 4, 7$ (9) by Lemma 8.6.

By Lemma 8.6, one of $d_5(v_2^{3k+2+3m}\psi_0)$ for $m = 0, 1, 2$ is zero. Suppose now that $d_5(v_2^{3k+2+3m}\psi_0) = 0$. Corollary 8.5 and (9.12) show $d_5(v_2^{3k+3+3m}\psi_0) = 0$, which contradicts the previous result if $m = 0, 1$. Therefore, $d_5(v_2^{3k+8}\psi_0) = 0$. With Lemma 8.6, we obtain the first four equations.

As we have seen above, $v_2^{3k+j}\psi_0$ is permanent by Remark 9.8 if $j \equiv 0, 4, 8$ (9). $v_2^{3l}\zeta_2$ is also permanent by Proposition 9.9. We then apply Lemma 1.9 to show

$$d_5(v_2^{3k+3l+j}\psi_0\zeta_2) = i_{1*}(\omega(v_2^{3k+j}\psi_0))\delta(v_2^{3l}\zeta_2) = 0.$$

In fact, $\delta(v_2^{3l}\zeta_2) = 0$ and $i_*\partial(v_2^{3k+3l+j}\psi_0\zeta_2) = v_2^{3k+3l}\zeta_2\delta(v_2^j\psi_0) \neq 0$ for $j \equiv 0, 4$ (9) by (9.11) and Proposition 5.9.

By Lemma 8.7, one of $d_5(v_2^{3k+3l+3m+2}\psi_0\zeta_2) = 0$ for $m = 0, 1, 2$. Then again apply Corollary 8.5 and (9.12) to see $d_5(v_2^{3k+3l+3m+3}\psi_0\zeta_2) = 0$, which contradicts the previous case if $m = 0, 1$, and so $m = 2$.

Lemmas 8.7 and 8.8 show the other equations. q.e.d.

By Lemma 9.3, we obtain

**Proposition 9.13.** 1) *The equations in Propositions* 9.9 *and* 9.10 *hold true in* $E_5^*(L_2V)$.

2) *For an equation* $d_5(x) = y$ *in Propositions* 8.4, 9.9 *and* 9.10, $d_5(xg_3) = yg_3$ *in* $E_5^*(L_2V_3)$.

To describe the $E_6$-terms, we introduce the notations:

$$K = \mathbf{Z}/3[v_2^9, v_2^{-9}], \quad V = \mathbf{Z}/3\{1, v_2, v_2^5\}, \quad \overline{V} = \mathbf{Z}/3\{v_2^2, v_2^3, v_2^4, v_2^6, v_2^7, v_2^8\},$$
$$P = \mathbf{Z}/3[b_{10}], \quad P_k = P/(b_{10}^k) = \mathbf{Z}/3[b_{10}]/(b_{10}^k),$$
$$F_1 = \mathbf{Z}/3\{1, v_2^3h_{10}, v_2^{-2}h_{11}, v_2^3b_{11}, v_2^{3k-1}\psi_0, v_2^{3k+3}\psi_1, v_2^{3k+2}\xi, v_2^{3k-4}\xi b_{11}\},$$
$$F_2 = \mathbf{Z}/3\{v_2^{-2}h_{11}, v_2^{3k-4}\xi b_{11}\} \quad \text{and} \quad F_3 = \mathbf{Z}/3\{v_2^3h_{10}, v_2^{3k+2}\xi\},$$

for $k$ and $l$ in $\mathbf{Z}/3$ given in Propositions 9.9 and 9.10.

Then Propositions 8.4, 9.9, 9.10 and 9.13 imply the following

**Proposition 9.14.** *The* $E_6$*-term of the Adams-Novikov spectral sequence for* $L_2V_3$ *are isomorphic to the tensor product of the* $K$*-module* $\Lambda(v_2^{3l}\zeta_2, g_3)$ *and the* $K$*-module of the direct sum of*

$$K \otimes V \otimes P \otimes F_1, \quad K \otimes \overline{V} \otimes P_2 \otimes F_2 \quad \textit{and}$$
$$K \otimes \overline{V} \otimes P_3 \otimes F_3.$$

*Here* $l \in \mathbf{Z}/3$ *is the element given in Proposition* 9.9.

Now Lemmas 9.4 and 9.7 show

**Theorem 9.15.** *The homotopy groups of $L_2V_3$ is isomorphic to the tensor product of $K$-module $\Lambda(v_2^{3l}\zeta_2)$ and the $K$-module of the direct sum of*

$$K\otimes V\otimes P_3\otimes F_1,\quad K\otimes\overline{V}\otimes P_2\otimes F_2\otimes\Lambda(g_3)\quad \textit{and}$$
$$K\otimes\overline{V}\otimes P_3\otimes F_3\otimes\Lambda(g_3).$$

*Proof.* The module structure of $E_7(L_2V_3)$ is read off from Proposition 9.14 and Lemma 9.4, which gives the $E_\infty$-term by Lemma 9.7. Since $V_3$ is an $M$-module spectrum, the extensions are trivial. q.e.d.

Propositions 8.4 and 9.13 also show

**Proposition 9.16.** *The $E_6$-term of the Adams-Novikov spectral sequence for $L_2V$ is isomorphic to the tensor product of the $K$-module $\Lambda(v_2^{3l}\zeta_2)$ and the $K$-module of the direct sum of*

$$K\otimes V\otimes P\otimes F_1,\quad K\otimes\overline{V}\otimes P_2\otimes F_2\quad \textit{and}\quad K\otimes\overline{V}\otimes P_3\otimes F_3.$$

## 10. The $E_{10}$-term of the Adams-Novikov spectral sequence for $L_2V$

In this section we compute $d_9 : E_6^s(L_2V)\to E_6^{s+9}(L_2V)$, in fact, $E_6^s(L_2V)=E_9^s(L_2V)$ by degree reason.

**Lemma 10.1.** *In the Adams-Novikov spectral sequence for computing $\pi_*(L_2V)$,*

$$\begin{aligned} d_9(v_2^jh_{10}) &= v_2^{j-3}b_{10}^5, & j\equiv 3,4,8\ (9), \quad \textit{and}\\ d_9(v_2^{3l+j}h_{10}\zeta_2) &= v_2^{3l+j-3}b_{10}^5\zeta_2, & j\equiv 3,4,8\ (9),\end{aligned}$$

*up to sign, for $l$ of Proposition* 9.9.

*Proof.* In this proof, the equations are all up to sign. Since $\beta_1^6=0$ in $\pi_*(S^0)$ (*cf.* [20]),

(10.2)
$$xb_{10}^6 \text{ is killed in } E_r^*(L_2V) \text{ for any permanent cycle } x\in E_2^*(L_2V).$$

As we have seen that $v_2^j$ is a permanent cycle for $j\equiv 0,1,5$ (9) in Theorem 2.6, $v_2^jb_{10}^6$ must be killed by virtue of (10.2). Comparing the degrees in sight of Theorem 5.8, either

$$d_9(v_2^{j+3}\psi_0)=v_2^jb_{10}^6 \quad\text{or}\quad d_9(v_2^{j+3}h_{10}b_{10})=v_2^jb_{10}^6$$

holds. If $j\equiv 0$ (9) for $k=0$, if $j\equiv 1$ or 5 (9) for $k=1$, or if $k=2$, then $v_2^{j+3}\psi_0$ dies in $E_9^*(L_2V)$ by Proposition 9.10, where $k$ is an element of $\mathbf{Z}/3$ given in Proposition 9.10. Therefore, the only choice is $d_9(v_2^{j+3}h_{10}b_{10})=v_2^jb_{10}^6$ in this case.

Suppose the other case where $j\equiv 1$ or 5 (9) if $k=0$, and $j\equiv 0$ (9) if $k=1$. Suppose further that $d_9(v_2^{j+3}\psi_0)=v_2^jb_{10}^6$, and multiply it by $h_{10}$ to get $d_9(v_2^{j+3}\psi_0h_{10})=v_2^jh_{10}b_{10}^6$, which equals

$$d_9(v_2^{j+3}\xi b_{10})=v_2^jh_{10}b_{10}^6$$

by the relation $\psi_0h_{10}=-b_{10}\xi$ of Proposition 5.9. Since $d_5(v_2^jb_{11})=\pm v_2^jh_{10}b_{10}^3$ by Proposition 8.4, we see that $d_9(v_2^{j+3}\xi b_{10})=v_2^jh_{10}b_{10}^6=d_5(v_2^jb_{11}b_{10}^3)=0$ in $E_9^*(L_2V)$. Therefore, $d_9(v_2^{j+3}\xi)=0$ by Lemma 7.1. If $d_{13}(v_2^{j+3}\xi)\neq 0$, then $d_{13}(v_2^{j+3}\xi)=v_2^{j-1}b_{10}^7\zeta_2$ by Theorem 5.8, since it is the only choice. Suppose that

$v_2^j b_{10}^7 \neq 0$ in $E_{13}^{15}(L_2V)$. This implies that $v_2^{j-1}\zeta_2$ is a permanent cycle by Lemma 7.1, and so $v_2^{j-1}b_{10}^6\zeta_2$ must be killed by virtue of (10.2). This is a contradiction. In fact, if $d_r(x) = v_2^{j-1}b_{10}^6\zeta_2$, then $d_r(xb_{10}) = v_2^{j-1}b_{10}^7\zeta_2$. By the supposition, $r \geq 13$, and there is no candidate for $x$ by Theorem 5.8. Therefore, $d_{13}(v_2^{j+3}\xi) = 0$. Again use (10.2) to see that $v_2^{j+3}\xi b_{10}^6$ must be killed in the spectral sequence for $\overline{E}_{19} \wedge V$. Theorem 5.8 and Lemma 6.9 show that the only possibility is

$$d_{13}(v_2^{j+7}h_{10}) = v_2^{j+3}b_{10}^6\xi.$$

Theorem 5.8 also shows that $v_2^{j+3}b_{10}^9\xi$ is not killed by $d_9$. Therefore the above equation implies

$$d_{13}(v_2^{j+7}h_{10}b_{10}^3) = v_2^{j+3}b_{10}^9\xi \quad \text{in} \quad E_{13}^{20}(L_2V),$$

which contradicts to Proposition 9.16 in which $v_2^{j+7}h_{10}b_{10}^3$ is shown not to be $E_6$-term for $j \equiv 0, 1, 5\ (9)$. Therefore, the only choice is also $d_9(v_2^{j+3}h_{10}b_{10}) = v_2^j b_{10}^6$ in this case, which shows the first equation.

By the same argument as above, we obtain the other ones. q.e.d.

**Corollary 10.3.** *For the differential $d_9$,*

$$\begin{aligned} d_9(v_2^j h_{11}) &= v_2^{j-4}b_{11}b_{10}^4, & j &\equiv 3,7,8\ (9), \quad \textit{and} \\ d_9(v_2^{j+3l}h_{11}\zeta_2) &= v_2^{j+3l-4}b_{11}b_{10}^4\zeta_2, & j &\equiv 3,7,8\ (9). \end{aligned}$$

*Proof.* Since $v_2^3b_{11}$ is a permanent cycle and originates from the sphere, the first equation of Lemma 10.1 with $v_2^3b_{11}$ gives rise to $d_9(v_2^{j+3}h_{10}b_{11}) = v_2^j b_{11}b_{10}^5$, and the first one since $h_{10}b_{11} = -v_2h_{11}b_{10}$ in Proposition 5.9. By the same reason, we obtain the second half. q.e.d.

**Corollary 10.4.** *The element $l$ of $\mathbf{Z}/3$ in Proposition 9.9 is zero. That is, $d_5(v_2^j\zeta_2) = 0$ for $j \equiv 0, 1, 5\ (9)$.*

*Proof.* Since $v_2h_{11}\zeta_2$ is a permanent cycle by Lemma 2.4, Corollary 10.3 indicates that $j \not\equiv 3, 7, 8\ (9)$ if $j + 3l \equiv 1\ (9)$. Therefore, $l \neq 1$. Similarly, $v_2^4h_{11}\zeta_2$ is permanent, and so $j \not\equiv 3, 7, 8\ (9)$ if $j + 3l \equiv 4\ (9)$. This shows that $l \neq 2$. q.e.d.

**Proposition 10.5.** *In the Adams-Novikov spectral sequence for $\pi_*(L_2V)$,*

$$\begin{aligned} d_9(v_2^j h_{10}) &= v_2^{j-3}b_{10}^5, & j &\equiv 3,4,8\ (9), \\ d_9(v_2^j h_{11}) &= v_2^{j-4}b_{11}b_{10}^4, & j &\equiv 3,7,8\ (9), \\ d_9(v_2^{j+3k}\xi) &= v_2^{j+3k-3}\psi_0 b_{10}^4, & j &\equiv 2,3,7\ (9), \quad \textit{and} \\ d_9(v_2^{j+3k}b_{11}\xi) &= v_2^{j+3k-2}\psi_1 b_{10}^5, & j &\equiv 1,5,6\ (9); \quad \textit{and} \end{aligned}$$

$$\begin{aligned} d_9(v_2^j h_{10}\zeta_2) &= v_2^{j-3}b_{10}^5\zeta_2, & j &\equiv 3,4,8\ (9), \\ d_9(v_2^j h_{11}\zeta_2) &= v_2^{j-4}b_{11}b_{10}^4\zeta_2, & j &\equiv 3,7,8\ (9), \\ d_9(v_2^{j+3k}\xi\zeta_2) &= v_2^{j+3k-3}\psi_0 b_{10}^4\zeta_2, & j &\equiv 2,3,7\ (9), \quad \textit{and} \\ d_9(v_2^{j+3k}b_{11}\xi\zeta_2) &= v_2^{j+3k-2}\psi_1 b_{10}^5\zeta_2, & j &\equiv 1,5,6\ (9) \end{aligned}$$

*up to sign for $k \in \mathbf{Z}/3$ in Proposition 9.10.*

*Proof.* We have seen the first, the second, the fifth and the sixth equations by Lemma 10.1 and Corollary 10.3.

By Theorem 5.8, for $j = 3k + m$ with $m \equiv 0, 4, 8$ (9),

$$d_9(v_2^j\psi_0) \in \mathbf{Z}/3\{v_2^{j-3}b_{10}^6, v_2^{j-3}h_{11}b_{10}^5\zeta_2, v_2^{j-3}\psi_1 b_{10}^4\zeta_2\} \quad \text{and}$$
$$d_9(v_2^j\psi_0\zeta_2) \in \mathbf{Z}/3\{v_2^{j-3}b_{10}^6\zeta_2\}.$$

By Proposition 9.16 and the first and the fifth equations, we see that $v_2^{j-3}b_{10}^6$ and $v_2^{j-3}b_{10}^6\zeta_2$ are killed by $d_5$ or $d_9$ for any $j$. The element $v_2^{j-3}h_{11}b_{10}^5\zeta_2$ is similarly seen to be away from the $E_{10}$-term. The other element $v_2^{j-3}\psi_1 b_{10}^4\zeta_2$ is also zero in the $E_6$-term by Proposition 9.16 in this case. Therefore, $d_9(v_2^j\psi_0) = 0$ and $d_9(v_2^j\psi_0\zeta_2) = 0$. Hence, $v_2^j\psi_0$ and $v_2^j\psi_0\zeta_2$ are permanent cycles of the Adams-Novikov spectral sequences for $\pi_*(V \wedge \overline{E}_{16})$ and $\pi_*(V \wedge \overline{E}_{17})$, respectively, and $v_2^j\psi_0 b_{10}^6$ and $v_2^j\psi_0 b_{10}^6\zeta_2$ must be killed by virtue of (10.2). By Lemma 6.9, $v_2^j h_{11}\zeta_2$ cannot be a killer of $v_2^j\psi_0 b_{10}^6$. Proposition 9.16 says that $v_2^{j+4}b_{10}$ and $v_2^{j+4}b_{10}\zeta_2$ are dead in the $E_6$-term for $j = 3k + m$ with $m \equiv 0, 4, 8$ (9). Therefore, we have the only choice $d_9(v_2^{j+3}\xi b_{10}^2) = v_2^j\psi_0 b_{10}^6$ and $d_9(v_2^{j+3}\xi b_{10}^2\zeta_2) = v_2^j\psi_0 b_{10}^6\zeta_2$, and so

$$d_9(v_2^{j+3}\xi) = v_2^j\psi_0 b_{10}^4 \quad \text{and} \quad d_9(v_2^{j+3}\xi\zeta_2) = v_2^j\psi_0 b_{10}^4\zeta_2$$

for $j \equiv 3k + m$ with $m \equiv 0, 4, 8$ (9) by Proposition 9.16. Now $\beta_{6/3} = v_2^3 b_{11}$ sends these to $d_9(v_2^{j+6}b_{11}\xi) = v_2^{j+3}b_{11}\psi_0 b_{10}^4 = v_2^{j+4}\psi_1 b_{10}^5$ and $d_9(v_2^{j+6}b_{11}\xi\zeta_2) = v_2^{j+4}\psi_1 b_{10}^5\zeta_2$ as desired. q.e.d.

These imply the following

**Theorem 10.6.** *The $E_{10}$-term of the Adams-Novikov spectral sequence for $\pi_*(L_2V(1))$ is isomorphic to the tensor product of $\Lambda(\zeta_2)$ and the direct sum of $P$-modules*

$$\begin{gathered} K \otimes V \otimes P_4 \otimes \{v_2^3 b_{11}, v_2^{3k-1}\psi_0\}, \\ K \otimes V \otimes P_5 \otimes \{1, v_2^{3k+3}\psi_1\}, \\ K \otimes \overline{V} \otimes P_2 \otimes \{v_2^{-2}h_{11}, v_2^{3k-4}\xi b_{11}\} \quad \textit{and} \\ K \otimes \overline{V} \otimes P_3 \otimes \{v_2^3 h_{10}, v_2^{3k+2}\xi\}. \end{gathered}$$

**Corollary 10.7.** $\pi_*(L_2V) \cong E_{10}^*(L_2V)$.

*Proof.* Theorem 10.6 shows $E_{10}^s(L_2V) = 0$ if $s > 12$, and we see that $d_{13} = 0$. Therefore, $E_\infty^*(L_2V) = E_{10}^*(L_2V)$. Since $L_2V$ is an $M$-module spectrum (*cf.* [27]), the homotopy groups of $L_2V$ are a $\mathbf{Z}/3$-vector space, and so there arises no extension problem, and we have the corollary. q.e.d.

## 11. The non-existence of $\beta$-elements

The definition (2.1) of $\beta$-elements gives us a representative of each $\beta$-element in the cobar complex, which we have in [14, Lemma 4.4]:

$$\beta_{3k+1} = [-v_2^{3k}b_{10} + \cdots] \quad \text{and} \quad \beta_{3k+2} = [-v_2^{3k}(t_2 \otimes t_1^3 - t_1 \otimes t_1^6) + \cdots].$$

Consider the composition

$$f : S^0 \xrightarrow{\eta} L_2S^0 \xrightarrow{L_2 i_1 i} L_2V(1),$$

where $\eta : S^0 \to L_2S^0$ is the localization map, and $i$ and $i_1$ are maps of (1.5) and (1.6).

**Proposition 11.1.** *In the Adams-Novikov spectral sequence for computing $\pi_*(S^0)$, $\beta_t \in E_2^2(S^0)$ dies under $d_5$ if $t \equiv 4, 7$ (9), and under $d_9$ if $t \equiv 8$ (9).*

*Proof.* By (2.1), $f_*(\beta_{3s+1}) = v_2^{3s}b_{10}$ and $f_*(\beta_{9s+8}) = v_2^{9s+7}h_{11}\zeta_2$. Then for the case where $t \equiv 4, 7$ (9), we compute $f_*(d_5(\beta_{3s+1})) = d_5(v_2^{3s})b_{10}$ by the naturality, which is not zero by Proposition 8.4 unless $s \equiv 0$ (3). Therefore, $d_5(\beta_{3s+1}) \neq 0$ in this case. In the same way, $f_*(d_9(\beta_{9s+8})) = d_9(v_2^{9s+7}h_{11}\zeta_2) \neq 0$ by Corollaries 10.3 and 10.4. q.e.d.

We next consider more $\beta$-elements. Recall [12] the definition of $\beta$-elements in the $E_2$-term of the Adams-Novikov spectral sequence for $S^0$. In $v_2^{-1}BP_*$, Miller, Ravenel and Wilson introduce elements $x_i$ such that

$$d(x_i) \equiv \pm v_1^{4\cdot 3^{i-1}-1}v_2^{2\cdot 3^{i-1}}t_1 \mod (3, v_1^{4\cdot 3^{i-1}}) \tag{11.2}$$

for the differential $d : v_2^{-1}BP_* \to v_2^{-1}BP_*(BP)$ defined by $d = \eta_R - \eta_L$. These are restated as elements of $E(2)_*$:

$$\begin{aligned} &x_i = v_2^{3^i} \text{ (for } i \leq 1), \quad x_2 = v_2^9 - v_1^8 v_2^7 \quad \text{and} \\ &x_i = x_{i-1}^3 + v_1^{4\cdot(3^{i-1}-1)}v_2^{2\cdot 3^{i-1}+1} \text{ ( for } i \geq 3). \end{aligned} \tag{11.3}$$

By (11.2), we see that

$$x_i^s \in \mathrm{Ext}^0(A/(3, v_1^j))$$

for $j < 4 \cdot 3^{i-1}$. Now consider the connecting homomorphisms

$$\partial_j : \mathrm{Ext}^0(A/(3, v_1^j)) \to \mathrm{Ext}^1(A/(3)) \quad \text{and} \quad \partial : \mathrm{Ext}^1(A/(3)) \to \mathrm{Ext}^2(A)$$

associated to the short exact sequences $0 \to A/(3) \to A/(3) \to A/(3, v_1^j) \to 0$ and $0 \to A \to A \to A/(3) \to 0$. Then $\beta$-elements are defined by (*cf.* [12])

$$\beta_{3^i s/j} = \partial\partial_j(x_i^s) \in \mathrm{Ext}^2(A) \tag{11.4}$$

for $j < 4 \cdot 3^{i-1}$ and $s > 1$, and at $s = 1$,

$$\beta_{3^i/j} = \partial\partial_j(v_2^{3^i}) \in \mathrm{Ext}^2(A)$$

for $j \leq 3^i$. As usual, $\beta_s = \beta_{s/1}$. This gives us an idea to define another $\beta$-element:

$$\beta_{3^i s/j} = \partial\partial_j(v_2^{3^i s}) \in \mathrm{Ext}^2(A) \tag{11.5}$$

for $j \leq 3^i$ and $s > 0$. In order to distinguish these $\beta$-elements, we denote the $\beta$-elements of (11.4) by $\beta^L_{3^i s/j}$ for $j \leq 3^i$ and $s > 0$. Then the $\beta$-elements of (11.5) give rise to

$$f_*(\beta_{3^i s/3^i}) = \begin{cases} [sv_2^{3^3 b_{i-3,s}+6}b_{10}], & i \text{ is odd} > 1, \\ [sv_2^{3^2 b_{i-2,s}}b_{11}], & i \text{ is even} > 0. \end{cases}$$

Here $b_{i,s} = (3^i - 1)/4 + 3^i(s-1)$. Therefore, Proposition 8.4 shows

**Theorem 11.6.** *$\beta_{9t+3/3}$ and $\beta_{3^i s/3^i}$ die under $d_5$ in $E_5^*(S^0)$ for $t \geq 0$, $i > 1$ and $s \not\equiv 0$ (3).*

In particular, we have Ravenel's odd primary Kervaire invariant theorem [18]:

**Corollary 11.7.** *In the Adams-Novikov spectral sequence at $p = 3$, $d_5(\beta_{3^i/3^i}) \neq 0$ for $i > 0$.*

An additional result is obtained from [18]

**Corollary 11.8.** *For the $\beta$-element $\beta_{9t/3,2} \in E_2^2(S^0)$,*

$$d_5(\beta_{9t/3,2}) \neq 0$$

*for $t \not\equiv 0$ (9).*

By (11.3), (11.4) and (11.5), we have a relation between these $\beta$-elements:

$$f_*(\beta^L_{3^is/j}) = f_*(\beta_{3^is/j}) - f_*(\beta_{(9s-2)\cdot 3^{i-2}/j-3^i+3^{i-2}})$$

for $j \leq 3^i$. Furthermore, (11.2) shows

$$f_*(\beta^L_{3^is/j}) = 0$$

for $i \geq 2$, $s > 0$ and $j \leq 3^i$. Therefore, we cannot tell from our results whether the three primary Kervaire invariant elements $b_{1i}$ of the Adams spectral sequence is permanent. In fact, for example, $b_{12}$ is obtained from $\beta_{9/9} \pm \beta_7 \in \pi_*(S^0)$, which is $\beta^L_{9/9}$ and $f_*(\beta^L_{9/9}) = 0 \in \pi_*(L_2V(1))$.

## References


[1] J. F. Adams, *Stable homotopy and generalised homology*, University of Chicago Press, Chicago, 1974. MR **53:**6534

[2] A. K. Bousfield, The localization of spectra with respect to homology, *Topology* **18** (1979), 257–281. MR **80m:**55006

[3] V. Gorbounov, S. Siegel and P. Symonds, The cohomology of the Morava stabilizer group $S_2$ at the prime 3, preprint.

[4] H-W. Henn, On the mod $p$ cohomology of profinite groups of positive $p$ rank, preprint.

[5] M. J. Hopkins and B. H. Gross, The rigid analytic period mapping, Lubin-Tate space, and stable homotopy theory, *Bull. Amer. Math. Soc.* **30** (1994), 76–86. MR **94k:**55009

[6] M. E. Mahowald, The order of the image of $J$ homomorphism, *Bull. AMS* **76** (1970), 1310–1313. MR **42:**5258

[7] M. E. Mahowald, The descriptive homotopy of the elements in the image of the $J$-homomorphism, Proc. Tokyo Conference on Manifolds (1973), 255–263. MR **51:**6804

[8] M. E. Mahowald, The image of $J$ in the $EHP$ sequence, *Ann. of Math.*, **116** (1982), 65–112. MR **83i:**55019

[9] M. Mahowald, D. Ravenel and P. Shick, The $v_2$-periodic homotopy of a certain Thom complex, preprint.

[10] M. Mahowald and K. Shimomura, The Adams-Novikov spectral sequence for the $L_2$-localization of a $v_2$ spectrum, *Contemporary Math.* **146** (1993), 237–250. MR **94g:**55012

[11] H. R. Miller, On relation between Adams spectral sequences, with an application to the stable homotopy of a Moore space, *J. of Pure and Appl. Algebra*, **20** (1981), 287–312. MR **82f:**55029

[12] H. R. Miller, D. C. Ravenel, and W. S. Wilson, Periodic phenomena in the Adams-Novikov spectral sequence, *Ann. of Math.* **106** (1977), 469–516. MR **56:**16626

[13] S. Oka, The stable homotopy groups of spheres II, *Hiroshima Math. J.*, **2** (1972), 99–161. MR **48:**1226

[14] S. Oka and K. Shimomura, On product of the $\beta$-elements in the stable homotopy of spheres, *Hiroshima Math. J.* **12**, (1982), 611–626. MR **84b:**55022

[15] S. Oka and H. Toda, 3-primary $\beta$-family in stable homotopy, *Hiroshima Math. J.* **5**, (1975), 447–460. MR **52:**6712

[16] S. Pemmaraju, private communication.

[17] D. C. Ravenel, The cohomology of the Morava stabilizer algebras, *Math. Z.*, **152** (1977), 287–297. MR **55:**4170

[18] D. C. Ravenel, The nonexistence of odd primary Arf invariant elements in stable homotopy, *Math. Proc. Cambridge Phil. Soc.* **83** (1978), 429–443. MR **57:**13938

[19] D. C. Ravenel, Localization with respect to certain periodic homology theories, *Amer. Math. J.*, **106** (1984), 351–414. MR **85k:**55009

[20] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Academic Press, 1986. MR **87j:**55003

[21] D. C. Ravenel, Progress report on the telescope conjecture, N. Ray and G. Walker, editors, *Adams Memorial Symposium on Algebraic Topology* **2**, 1–21, Cambridge Univ. Press, Cambridge, 1992. MR **94h:**55023

[22] D. C. Ravenel, *Nilpotence and Periodicity in Stable Homotopy Theory,* Ann. of Math. Stud. 128, Princeton University Press, Princeton, 1992. MR **94b:**55015

[23] K. Shimomura, 3-primary $\beta$-family in stable homotopy of a finite spectrum, *Hiroshima Math. J.* **26** (1996), 341–349. CMP 96:15

[24] K. Shimomura and A. Yabe, The homotopy group $\pi_*(L_2S^0)$, *Topology* **34** (1995), 261–289. MR **96b:**55015

[25] H. Toda, An important relation in homotopy groups of spheres, *Proc. Japan Acad.* **43** (1967), 839–842. MR **37:**5872

[26] H. Toda, On spectra realizing exterior parts of the Steenrod algebra, *Topology*, **10** (1971), 53–65. MR **42:**6814

[27] H. Toda, Algebra of stable homotopy of $\mathbf{Z}_p$-spaces and applications, *J. Math. Kyoto Univ.*, **11** (1971), 197–251. MR **45:**2708

[28] N. Yagita, Small subalgebras of Steenrod and Morava stabilizer algebras, preprint.

[29] R. S. Zahler, The Adams-Novikov spectral sequence for the spheres, *Ann. of Math.*, **96** (1972), 480–504. MR **47:**7742



Faculty of Education, Tottori University, Tottori, 680, Japan

*E-mail address*: `katsumi@fed.tottori-u.ac.jp`